FLUID LIMIT OF A HEAVILY LOADED EDF QUEUE WITH IMPATIENT CUSTOMERS

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Abstract. In this paper, we present the fluid limit of an heavily loaded Earliest Deadline First queue with impatient customers, represented by a measure-valued process keeping track of residual time-credits of lost and waiting customers. This fluid limit is the solution of an integrated transport equation. We then use this fluid limit to derive fluid approximations of the processes counting the number of waiting and already lost customers.

1. Introduction

Queueing theory is a keystone of the development of current telecommunications systems. Engineers now aim to guarantee the grade of service customers are entitled to receive according to their contracts with the carrier. One way to meet this objective is to schedule requests according to their "importance". Of the utmost interest, are the audio and video traffic flows, which are subject to severe transmission delay constraints. In particular, some requests can be thought as "impatient" since it may be better to discard some packets which would eventually arrive too late in order to favor some other packets which still can meet their delay requirements.

Another very active branch of queueing theory is, nowadays, devoted to the analysis of call-centers where customers are *impatient*: they tolerate to wait up to a certain limit upon which they depart from the queueing line hence are considered as lost both for the queueing system and for the called-service provider. It is then crucial to develop service disciplines which ensure a maximal number of served customers by controlling waiting times keeping them within impatience bounds. These disciplines are commonly referred to as real-time service disciplines.

In real-time queuing theory, each customer is not only identified by his arrival time and service duration but also by a *deadline*. This means that a customer has a given period of time (his time credit, i.e., the remaining time before his deadline) during which he should enter the service booth. This time credit decreases at unit rate as time goes on. If it expires before the customer enters service, the customer is either lost and the discipline is said to be hard, or he is kept in the waiting line and the discipline is said to be soft. The discipline we address here is the so-called Earliest-Deadline-First discipline in its "hard" version: the customer having the smallest time credit is served first and whenever the credit-time of a customer expires before it is served, this customer is lost.

To find which service discipline is best, one can compare them within a static scenario, i.e., customers to be served are all present at initial time and no new customer enter the system, service duration and impatience of each customer are

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all known at the beginning; or in dynamic environments, i.e., customers arrive randomly, their service duration and impatience are only known stochastically. In both settings, it appears that the so-called Earliest-Deadline-First (EDF for short) discipline is optimal. It is known for a while [Der74], that EDF discipline is optimal for the static approach: if any (real-time) service discipline can serve the customers of a given scenario without loss then EDF also does. Within random environments, it has been proven in [PT88] and generalized in [Moy05] that EDF discipline ensures the least possible failure probability, i.e., the least number customers lost by missing their deadline.

Yet, apart from the notable exception of deterministic deadlines for which EDF discipline reduces to the FIFO service policy with impatience, no closed form of the loss probability is known. The only satisfying quantitative approach so far consists in numerically assessing the loss probability for an EDF system with Markov-chain approximations [HXD88, PK91, ND92, PT88].

When no simple tractable object can describe a queueing system, one wants to identify its "mean behavior". One hopes that a Markovian process characterizing the system, when suitably normalized, can be approximated by a fluid limit that is, a deterministic continuous function of the time. Then the fluid limit describes the general behavior of the considered process. Numerous queueing systems have already been investigated this way (see, for instance, [Rob00, Bor67] for a pure delay system, [DLS01] for a soft EDF queue, [GPW01] for a queue run under a processor sharing service discipline). For instance, L_t being the amount of customers at time t in an $M/M/1$ queue with parameters λ and μ, one proves that the sequence of processes $\left\{\bar{L}^n\right\}_{n\in\mathbb{N}^*}$ defined by $\bar{L}_0^{(n)}=1$ and for all $t>0$, $\bar{L}_t^{(n)}:=n^{-1}L_{nt}$ tends in distribution to $\bar{L}:=((1+(\lambda-\mu)t)^+,\, t\in\mathbb{R}^+)$ and that $\left\{\sqrt{n}\left(\bar{L}^{(n)}-\bar{L}\right)\right\}_{n\in\mathbb{N}^*}$ converges in distribution to a diffusion process. The fluid approximation of the system presents the same first order characteristics as the "real" system: it fills in at velocity λ and empties at velocity μ, the congestion reaches 0 to the condition $\lambda<\mu$ (this is Loynes's stability condition) after a time $\lambda-\mu$ (mean duration of a busy period).

We want to obtain the same type of information for an $M/M/1$ queueing system with impatient customers. In this case, it is easily seen that the process $(X_t)_{t\geq 0}$ which counts the number of customers in the system is no longer Markovian. Indeed, the value of X_{t+h} not only depends on X_t, but also on all the time credits of all the X_t customers present in the queue at time t. Therefore we describe the system by the point measure-valued process $(\nu_t)_{t\geq 0}$ whose unit of mass are the time credits of all the customers waiting in the queue, or already discarded.

Formally, it is rather straightforward in our case to write down the infinitesimal generator of the Markov process $(\nu_t)_{t\geq 0}$, see Theorem 2 below. It is made of four terms, all but one are standard and represent the evolution of the process when an arrival, a departure or nothing occurs during an infinitesimal time period. The natural but unusual term is the term due to the continuous decreasing of the residual deadlines at unit rate as time goes on. This term involves a "spatial derivative" of the measure ν, a notion which can only be rigorously defined within the framework of distributions. Because of this term, the fluid limit equation (see (23)) is the integrated version of a partial differential equation rather than an ordinary differential equation as it is the rule in the previously studied queueing systems. Thus, the famous Gronwall's Lemma is of no use here. Fortunately, the partial differential

equation which pops up, known as transport equation, is simple enough to have a closed form solution – see Theorem 1. Thanks to that, we can then proceed as usual to show the strong convergence of the renormalized process to the fluid limit.

This paper is organized as follows. After some preliminaries, we define and solve the integrated transport equation in the space of tempered distributions. In Section 4, we establish that the above described process $(\nu_t)_{t\geq 0}$, is a weak Feller Markov process and give its infinitesimal generator. In Section 6, we prove the fluid limit theorem. The last section is devoted to applications to the EDF driven queue with deterministic initial time credits, and to a pure delay system.

2. Preliminaries

We denote by $\mathcal{D}_b$, respectively $\mathcal{C}_0$ and $\mathcal{C}_b$ the set of real-valued functions defined on $\mathbb{R}$ which are bounded, right-continuous with left-limits (rcll for short), respectively continuous vanishing at infinity and bounded continuous. The space $\mathcal{D}_b$ is equipped with the Skorokhod topology and $\mathcal{C}_0$ and $\mathcal{C}_b$ with the topology of the uniform convergence. The space of bounded differentiable functions from $\mathbb{R}$ to itself is denoted by $\mathcal{C}_b^1$ and for $\phi \in \mathcal{C}_b^1$, $\| \phi \|_\infty := \sup_{x\in\mathbb{R}}(|\phi(x)| + |\phi'(x)|)$. For all $f \in \mathcal{D}_b$ and all $x \in \mathbb{R}$, we denote by $\tau_x f$, the function $\tau_x f(.) := f(. - x)$.

The Schwartz space, denoted by $\mathcal{S}$, is the space of infinitely differentiable functions, equipped with the topology defined by the semi-norms:

$$| \phi |_{a,b} := \sup_{x\in\mathbb{R}} | x^a \frac{d^b}{dx^b}\phi(x) |, \; a \in \mathbb{N}, \; b \in \mathbb{N}.$$

Its topological dual, the space of tempered distributions, is denoted by $\mathcal{S}'$, and the duality product is classically denoted $\langle \mu, \phi \rangle$. The Fourier transform on $\mathcal{S}$ is defined by $\widehat{\phi}(\xi) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-i\xi x}\phi(x)\,dx$ and the Fourier transform is defined on $\mathcal{S}'$ by the duality relation $\langle \widehat{\mu}, \phi \rangle = \langle \mu, \widehat{\phi} \rangle$.

The set of finite positive measures on $\mathbb{R}$ is denoted by $\mathcal{M}_f^+$ and $\mathcal{M}_p$ is the set of finite counting measures on $\mathbb{R}$. The space $\mathcal{M}_f^+$ is embedded with the weak topology, $\sigma(\mathcal{M}_f^+, \mathcal{C}_b)$, for which $\mathcal{M}_f^+$ is Polish (we write $\langle \mu, f \rangle = \int f\,d\mu$ for $\mu \in \mathcal{M}_f^+$ and $f \in \mathcal{D}_b$). We also denote for all $x \in \mathbb{R}$ and all $\nu \in \mathcal{M}_f^+$, $\tau_x\nu$ the measure satisfying for all Borel set B, $\tau_x\nu(B) := \nu(B - x)$. Let $\mathcal{C}_0(\mathcal{M}_f^+, \mathbb{R})$, be the set of continuous functions from $\mathcal{M}_f^+$ to $\mathbb{R}$, vanishing at infinity, endpwed with the topology of the sup norm. Let $0 < T < \infty$, for E a Polish space, we denote $\mathcal{C}([0,T], E)$, respectively $\mathcal{D}([0,T], E)$, the Polish space (for its usual strong topology) of continuous, respectively rcll, functions from $[0,T]$ to E.

3. The integrated transport equation

The *transport equation* on $\mathcal{C}^1(\mathbb{R} \times \mathbb{R}_+, \mathbb{R})$ with unknown $u(x,t)$ is defined as:

$$\text{(E)} \qquad \begin{aligned} \partial_t u &= -b\partial_x u + f \text{ in } \mathbb{R} \times (0,\infty), \\ u &= h \text{ at } \mathbb{R} \times \{t = 0\}, \end{aligned}$$

where b is a real number, f is a function of $\mathcal{C}^1(\mathbb{R} \times \mathbb{R}+, \mathbb{R})$, and $h \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$. It is well known (see [Eva98]) that (E) admits a unique solution given for all x, t by:

$$(1) \qquad u(x,t) = h(x - tb) + \int_0^t f(x + (s - t)b, s)ds.$$

Let us define the following extension of the transport equation:

Definition 1. *Let $T>0$, $K \in \mathcal{S}'$, $(g_t)_{t\geq 0} \in \mathcal{D}([0,T],\mathcal{S}')$ such that $g_0 \equiv 0$ and b be a real number. The process $(\eta_t)_{t\geq 0}$ satisfies the integrated transport equation E(K,g,b) on $\mathcal{D}([0,T],\mathcal{S}')$ if for all $\phi \in \mathcal{S}'$, and for all $t \in [0,T]$:*

$$\mathrm{E}(K,g,b) \qquad \langle \eta_t, \phi\rangle = \langle K,\phi\rangle - b\int_0^t \langle \eta_s, \phi'\rangle\, ds + \langle g_t,\phi\rangle.$$

Theorem 1. *The integrated transport equation ($E(K,g,b)$) admits a unique solution $(L_t)_{t\geq 0}$ in $\mathcal{D}([0,T],\mathcal{S}')$, satisfying for all $\phi \in \mathcal{S}$ and for all $t\in[0,T]$:*

$$\langle L_t,\phi\rangle = \langle K, \tau_{bt}\,\phi\rangle + \langle g_t,\phi\rangle - b\int_0^t \langle g_s, \tau_{b(t-s)}\,\phi'\rangle\, ds. \tag{2}$$

Proof. Let L and M be two solutions of ($\mathrm{E}(K,g,b)$) and let $N = L-M$. For all $t\in[0,T]$, it follows from $\mathrm{E}(K,g,b)$ that for all $\phi \in \mathcal{S}$:

$$\frac{d}{dt}\langle \widehat{N_t},\phi\rangle = -b\langle N_t, \widehat{\phi}\,'\rangle.$$

Denoting for all $\xi\in\mathbb{R}$, $\psi(\xi) := -i\xi$, this can be rewritten:

$$\frac{d}{dt}\langle \widehat{N_t},\phi\rangle = -b\langle N_t,\widehat{\psi\phi}\rangle = -b\langle \widehat{N_t},\psi\phi\rangle = -b\langle \psi\widehat{N_t},\phi\rangle.$$

Solving the latter differential equation yields for all $\phi\in\mathcal{S}$ and all $t\in[0,T]$:

$$\langle \widehat{N_t},\phi\rangle = \langle \widehat{N_0}e^{b\psi t},\phi\rangle = \langle \widehat{N_0}, e^{b\psi t}\phi\rangle = \langle N_0, \widehat{e^{b\psi t}\phi}\rangle = 0,$$

hence for all $t\in[0,T]$, $N_t \equiv 0$. Therefore, there is at most one solution to ($\mathrm{E}(K,g,b)$).

The process $(L_t)_{t\geq 0}$ defined by (2) belongs to $\mathcal{D}([0,T],\mathcal{S}')$ and for $\phi\in\mathcal{S}$, we have:

$$\begin{aligned} b\int_0^t &\langle L_s,\phi'\rangle\, ds \\ &= b\int_0^t \langle K, \tau_{bs}\phi'\rangle\, ds + b\int_0^t\langle g_s,\phi'\rangle\, ds - b^2\int_0^t\int_0^s \langle g_r, \tau_{b(s-r)}\phi'\rangle\, dr\, ds. \end{aligned}$$

Since $\partial_t\langle \zeta, \tau_{bt}\phi\rangle = -b\langle\zeta,\tau_{bt}\phi'\rangle$, we get:

$$\begin{aligned} b\int_0^t \langle L_s,\phi'\rangle\, ds &= -\int_0^t \frac{d}{ds}\left(\langle K,\tau_{bs}\phi\rangle\right)\, ds + b\int_0^t \langle g_s,\phi'\rangle\, ds \\ &\qquad + b\int_0^t\int_r^t \frac{d}{ds}\left(\langle g_r, \tau_{b(s-r)}\phi'\rangle\right)\, ds\, dr \\ &= -\langle K,\tau_{bt}\phi\rangle + \langle K,\phi\rangle + b\int_0^t\langle g_s,\tau_{b(t-s)}\phi'\rangle\, ds = -\langle L_t,\phi\rangle + \langle K,\phi\rangle + \langle g_t,\phi\rangle, \end{aligned}$$

The process $(L_t)_{t\geq 0}$ thus satisfies ($\mathrm{E}(K,g,b)$). ☐

4. The profile process

Following Barrer's notation [Bar57], we throughout this paper consider a queueing system with impatient customers M/M/1/1+GI-EDF:

- customers arrive at times $\{T_i\}_{i\in\mathbb{N}^*}$. The process defined for all t by
$$N_t := \sum_{i\in\mathbb{N}^*} \mathbf{1}_{\{T_i\leq t\}}$$
is a Poisson process of intensity $\lambda > 0$,
- a first sequence of marks $\{\sigma_i\}_{i\in\mathbb{N}^*}$, the sequence of service durations requested by the customers, is i.i.d. with the distribution of σ which an exponentially distributed with parameter $\mu > 0$ random variable,
- the customers are impatient: i.e., the i-th customer leaves the system, and is lost forever, when he doesn't reach the service booth before his specific deadline, $T_i + D_i$. In other words, he is initially labelled with a random variable referred to as his patience, or initial time credit, D_i. The marks $\{D_i\}_{i\in\mathbb{N}^*}$ are independent and identically distributed with the distribution of D, an almost-surely non-negative and integrable random variable. The time credits of the customers decrease continuously with time, at velocity one (in time units). Provided that the i-th customer entered the system before t ($T_i \leq t$), but did not reach the service booth before t, we denote $D_i(t)$ the residual time credit at t of this customer, i.e., the residual time before his possible elimination. Therefore:
$$D_i(t) = D_i - (t - T_i),$$
and $D_i(t) \leq 0$ means that the i-th customer has been lost, reaching his patience before t before entering the service,
- there is 1 non idling server and a buffer of infinite capacity,
- the service discipline is EDF (i.e., Earliest Deadline First): when completing a service, the server deals with the customer whose residual time credit is the smallest among all the customers in the buffer, if any. This service then proceeds until completion, without any interruption.

Let us finally define the following performance processes:

$$\begin{aligned}
X_t &:= \text{ Number of customers in the system (buffer + service booth) at } t\ ,\\
Q_t &:= (X_t - S)^+ = \text{ Number of customers in the buffer at } t\ ,\\
\mathcal{S}_t &:= \text{ Number of customers served up to time } t\ ,\\
P_t &:= \text{ Number of customers lost up to time } t\ ,
\end{aligned}$$

At time t, provided that the buffer is non-empty, denote for $i = 1, ..., Q_t$, $R_i(t)$ the i-th residual time credit of a customer in the buffer at t, ranked in the increasing order:
$$R_1(t) < R_2(t) < .... < R_{Q_t}(t).$$
Provided that at least one customer has been lost at t ($P_t \neq 0$), for $i = 1, ..., P_t$, denote $R_{-i}(t)$, the i-th residual time credit among the customers lost up to t in the decreasing order:
$$R_{-P_t}(t) < R_{-P_t-1}(t) < .... < R_{-1}(t).$$

The time credit profile of the system at t is the following measure:

$$\nu_t := \sum_{i=1}^{Q_t} \delta_{R_i(t)} + \sum_{i=1}^{P_t} \delta_{R_{-i}(t)},$$

where δ_x is the Dirac mass at x. Provided that the buffer is non-empty at t, we denote for all $i = 1, ..., Q_t$,

$$t_i(\nu_t) := R_i(t),$$

the i-th point of ν_t (in the increasing order) on the positive half-line.

The service discipline can be represented as follows : when the server completes a service (say at time s), he first deals with the customer whose time credit is given at this time by:

$$t_1(\nu_s) = R_1(s) > 0,$$

provided that $Q_s \neq 0$. The customer corresponding to the atom $t_1(\nu_s)$, being chosen by the server, leaves the buffer: the corresponding atom $\delta_{t_1(\nu_s)}$ is erased from the point measure ν_t for all $t \geq s$ (this customer won't ever reappear in the buffer, since the service discipline is non-preemptive).

By profile process of the queue, we mean, the process $(\nu_t)_{t\geq 0}$ of the time credit profiles at t. This process is fully characterized by its initial value ν_0, the real numbers $\lambda > 0$ and $\mu > 0$ and the non negative integrable random variable D. This process will consequently be referred to as the profile process associated to (ν_0, λ, μ, D). The dynamics of the profile process can be depicted as follows. The atoms are translated continuously towards left at velocity 1, at the arrival time T_i, an atom is added to the measure ν_{T_i} at D_i the initial time credit of the arriving customer, and at an end of service $\tilde{T}_i$, an atom disappear from the measure $\nu_{\tilde{T}_i}$ at $t_1(\nu_{\tilde{T}_i})$. Figure 1 shows a typical path of the profile process. Note, that the buffer congestion and loss processes can be deduced from the profile process by writing for all $t \geq 0$:

$$Q_t = \langle \nu_t, \mathbf{1}_{\mathbb{R}_+^*} \rangle,\ P_t = \langle \nu_t, \mathbf{1}_{\mathbb{R}-} \rangle,$$

since the waiting, resp. already lost, customers at t are those who have positive, resp. non positive, time credits at t.

5. Markov property

Denote for all t, A_t the remaining time before the next arrival after t, and for all t such that $X_t > 0$, F_t the remaining time before the next end of service after t. For all $t, h > 0$:

$$\nu_{t+h} = \begin{cases} \tau_h \nu_t & \text{if } A_t > h \text{ and } F_t > h, \text{ or } Q_t = 0, \\ \tau_h \nu_t - \delta_{t_1(\nu_t)-h} & \text{if } A_t > h,\ F_t < h \text{ and } Q_t > 0, \\ \tau_h \nu_t + \delta_{d_k-(t+h-t_k)} & \text{if } A_t < h \text{ and } F_t > h \text{ or } Q_t = 0 \\ & \text{and the customer arrives at } t_k \\ & \text{affected with the initial time credit } d_k, \end{cases}$$

the more complex events (several arrivals, several ends of service, or arrivals and ends of service) between t and $t+h$ being of probability $o(h)$. This dynamics shows in particular that $(\nu_t)_{t\geq 0} \in \mathcal{D}\left([0,\infty), \mathcal{M}_f^+\right)$, since $(\langle \nu_t, \phi \rangle, t \geq 0)$ belongs to $\mathcal{D}([0,\infty), \mathbb{R})$ for all $\phi \in \mathcal{C}_b$. We finally define the filtration:

$$\mathcal{F}_t := \sigma\left(\nu_s(B), s \leq t, B \in \mathfrak{B}(\mathbb{R})\right).$$

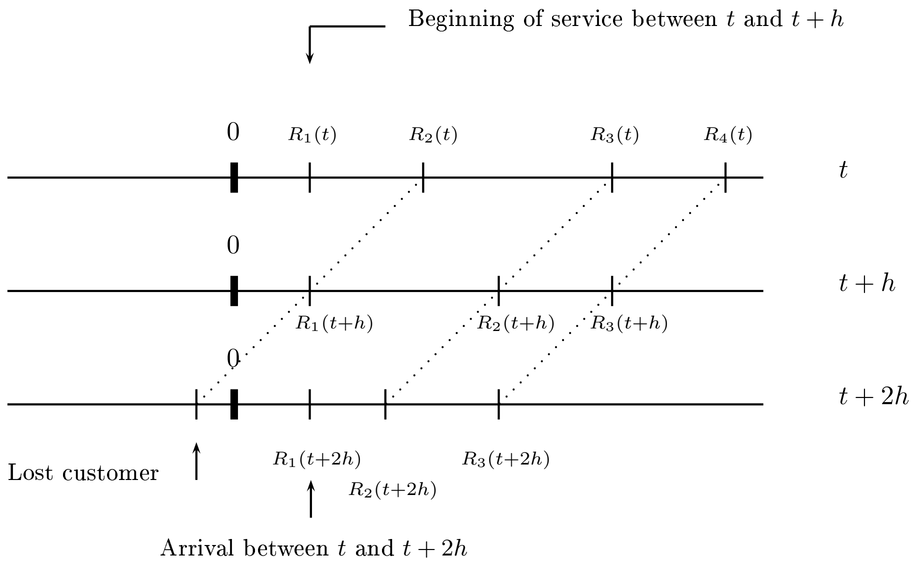

FIGURE 1. Dynamics of the profile process.

Theorem 2. *The profile process* $(\nu_t)_{t\geq 0}$ *associated to* (ν_0, λ, μ, D) *is a weak Feller process with respect to* $(\mathcal{F}_t)_{t\geq 0}$*, whose infinitesimal generator is given by:*

$$
\begin{aligned}
(3) \quad \mathcal{A}F(\nu) = \lim_{h\to 0} \frac{F(\tau_h\nu) - F(\nu)}{h} &- \left(\lambda + \mu \mathbf{1}_{\left\{\nu_t(\mathbb{R}_+^*)>0\right\}}\right) F(\nu) \\
&+ \mu F\left(\nu - \delta_{t_1(\nu)}\right) \mathbf{1}_{\left\{\nu(\mathbb{R}_+^*)>0\right\}} + \lambda \int F\left(\nu + \delta_d\right) \, d\mathbf{P}_D(d),
\end{aligned}
$$

for all F *in the domain of* $\mathcal{A}$*:*

$$
\mathcal{D}(\mathcal{A}) := \mathcal{C}_0(\mathcal{M}_f, \mathbb{R}) \cap \left\{ \lim_{h\to 0} \frac{F(\tau_h.) - F}{h} \ \textit{exists} \right\}.
$$

Proof. For all $t, h \geq 0$ and all bounded measurable function $F : \mathcal{M}_f^+ \to \mathbb{R}$:

$$
\begin{aligned}
(4) \quad \mathbf{E}\left[F(\nu_{t+h}) | \mathcal{F}_t\right] = &\left(1 - \left(\lambda + \mu \mathbf{1}_{\left\{\nu_t(\mathbb{R}_+^*)>0\right\}}\right) h\right) F(\tau_h \nu_t) \\
&+ \mu h F\left(\tau_h \nu_t - \tau_h \delta_{t_1(\nu_t)}\right) \mathbf{1}_{\left\{\nu_t(\mathbb{R}_+^*)>0\right\}} \\
&+ \lambda h \int F\left(\tau_h \nu_t + \tau_h \delta_d\right) \, d\mathbf{P}_D(d) + o(h) \\
&=: T_h F(\nu_t).
\end{aligned}
$$

Thus, according to [Daw93], p.18, $(\nu_t)_{t\geq 0}$ is a weak homogeneous Markov process, whose transition function is given by $(T_h, h \geq 0)$. For $F \in \mathcal{C}_0\left(\mathcal{M}_f^+, \mathbb{R}\right)$, it is easily seen from (4), that $T_h F \in \mathcal{C}_0\left(\mathcal{M}_f^+, \mathbb{R}\right)$ for all $h \geq 0$. Since $\mathcal{M}_f^+$ embedded with

the weak topology is locally compact separable, it routinely follows that $(\nu_t)_{t\geq 0}$ is a weak Feller process whose infinitesimal generator of ν is given by (3). $\square$

Corollary 1. *For all $\phi \in \mathcal{C}_b^1$, the process defined for all $t \geq 0$ by:*

$$\begin{aligned}(5)\quad M_\phi(t) = \langle \nu_t, \phi\rangle - \langle \nu_0, \phi\rangle - \int_0^t \langle \nu_s, \phi'\rangle\, ds \\ + \mu \int_0^t \phi\left(t_1(\nu_s)\right) \mathbf{1}_{\left\{\nu_s(\mathbb{R}_+^*)>0\right\}}\, ds - \lambda t \mathbf{E}\left[\phi(D)\right]\end{aligned}$$

is an rcll $\mathcal{F}_t$-martingale such that $M_\phi(t) \in L^2$ for all $t \geq 0$. Its increasing process is given for all $t \geq 0$ by:

$$(6)\qquad <M_\phi>_t = \mu \int_0^t \phi^2\left(t_1(\nu_s)\right) \mathbf{1}_{\left\{\nu_s(\mathbb{R}_+^*)>0\right\}}\, ds + \lambda t \mathbf{E}\left[\phi^2(D)\right].$$

Proof. Let $\phi \in \mathcal{C}_b^1$. Define the mapping $\Pi_\phi : \mathcal{M}_f^+ \mapsto \mathbb{R}$ for all ν by:

$$\Pi_\phi(\nu) := \langle \nu, \phi\rangle.$$

Since

$$(7)\qquad \lim_{h\to 0} \frac{1}{h}\Big(\Pi_\phi(\tau_h \nu) - \Pi_\phi(\nu)\Big) = -\langle \nu, \phi'\rangle,$$

we have for all $\nu \in \mathcal{M}_p$:

$$\mathcal{A}\Pi_\phi(\nu) = -\langle \nu, \phi'\rangle - \mu\phi\left(t_1(\nu)\right) \mathbf{1}_{\left\{\nu(\mathbb{R}_+^*)>0\right\}} + \lambda \mathbf{E}\left[\phi(D)\right].$$

Furhtermore,

$$\lim_{h\to 0} \frac{1}{h}\Big(\Pi_\phi(\tau_h \nu)^2 - \Pi_\phi(\nu)^2\Big) = -2\langle \nu, \phi\rangle\langle \nu, \phi'\rangle,$$

and hence for all $\nu \in M_p$:

$$\mathcal{A}\Pi_\phi^2(\nu) = 2\langle \nu, \phi\rangle \mathcal{A}\Pi_\phi(\nu) + \mu\phi^2\left(t_1(\nu)\right) \mathbf{1}_{\left\{\nu(\mathbb{R}_+^*)>0\right\}} + \lambda \mathbf{E}\left[\phi^2(D)\right].$$

From Dynkin's lemma [EK86, Dyn65], it follows that M_ϕ and the process defined for all t by

$$N_\phi(t) = \Pi_\phi^2(\nu_t) - \Pi_\phi^2(\nu_0) - \int_0^t \mathcal{A}\Pi_\phi^2(\nu_s)\, ds$$

are $\mathcal{F}_t$-local martingales. This entails that $<M_\phi>_t = <\langle \nu_., \phi\rangle>_t$ for all $t \geq 0$, and thus, Itô's integration by parts formula yields to:

$$\begin{aligned}\langle \nu_t, \phi\rangle^2 = \langle \nu_0, \phi\rangle^2 + 2\int_0^t \langle \nu_s, \phi\rangle\, dM_\phi(t) \\ + 2\int_0^t \langle \nu_s, \phi\rangle \mathcal{A}\Pi_\phi(\nu_s)\, ds + <\langle \nu_., \phi\rangle>_t.\end{aligned}$$

Hence, for all $t \geq 0$:

$$\begin{aligned}2\int_0^t \langle \nu_s, \phi\rangle\, dM_\phi(t) + <\langle \nu_., \phi\rangle>_t \\ = N_\phi(t) + \mu \int_0^t \phi^2\left(t_1(\nu_s)\right) \mathbf{1}_{\left\{\nu_s(\mathbb{R}_+^*)>0\right\}}\, ds + \lambda t \mathbf{E}\left[\phi^2(D)\right],\end{aligned}$$

and by identifying the finite variation processes, we obtain (6). $\square$

6. Fluid limit

For all $n \in \mathbb{N}^*$, denote by ν_0^n, a measure on $\mathbb{R}_+^*$, and define $(\nu_t^n)_{t\geq 0}$, the profile process of the M/M/1/1+GI-EDF queue whose initial state is represented by the profile ν_0^n, whose arrival process $(N_t^n)_{t\geq 0}$ is Poisson of intensity $\lambda^n >$, where the customers request service durations are exponentially distributed of mean expectation $(\mu^n)^{-1}$, and have initial time credits i.i.d. with the distribution of D^n. The process $(\nu_t^n)_{t\geq 0}$ is in other words the profile process associated to $(\nu_0^n, \lambda^n, \mu^n, D^n)$. Also denote $(\mathcal{F}_t^n, t \geq 0)$, the associated filtration,

$$\tau_0^n := \inf\left\{t \geq 0, \nu_t^n\left(\mathbb{R}_+^*\right) = 0\right\},$$

the first time when the buffer is empty, and

$$\omega_0^n := \inf\left\{t \geq 0, t_1(\nu_t^n) = 0\right\},$$

the first time of loss of the system. We also define as previously the performance processes of the n-th system: $(X_t^n)_{t\geq 0}$, $(Q_t^n)_{t\geq 0}$, given for all t by $Q_t^n = \langle \nu_t^n, \mathbf{1}_{\mathbb{R}_+^*}\rangle$, $(\mathcal{S}_t^n)_{t\geq 0}$, $(P_t^n)_{t\geq 0}$, given by $P_t^n = \langle \nu_t^n, \mathbf{1}_{R-}\rangle$.
According to Theorem 1, for all $\phi \in \mathcal{C}_b^1$, the process defined for all $t \geq 0$ by:

$$\begin{aligned}(8)\quad M_\phi^n(t) = \langle \nu_t^n, \phi\rangle - \langle \nu_0^n, \phi\rangle - \int_0^t \langle \nu_s^n, \phi'\rangle\, ds \\ + \mu^n \int_0^t \phi\left(t_1(\nu_s^n)\right) \mathbf{1}_{\left\{\nu_s^n(\mathbb{R}_+^*)>0\right\}}\, ds - \lambda^n t \mathbf{E}\left[\phi(D^n)\right]\end{aligned}$$

is an rcll $\mathcal{F}_t^n$-martingale, such that $M_\phi^n(t) \in L^2$ for all t, and whose increasing process is given for all $t \geq 0$ by:

$$(9)\qquad < M_\phi^n >_t = \mu^n \int_0^t \phi^2\left(t_1(\nu_s^n)\right) \mathbf{1}_{\left\{\nu_s^n(\mathbb{R}_+^*)>0\right\}}\, ds + \lambda^n t \mathbf{E}\left[\phi^2(D^n)\right].$$

We normalize the process $(\nu_t^n)_{t\geq 0}$ in time, space and weight the following way: for all Borel set B and for all t, define

$$\bar{\nu}_t^n(B) = \frac{\nu_{nt}^n(nB)}{n},$$

where

$$nB := \{nx, x \in B\}.$$

The first positive atom of $\bar{\nu}_t^n$ is therefore given by:

$$t_1(\bar{\nu}_t^n) = \frac{t_1\left(\nu_{nt}^n\right)}{n}.$$

We also denote $(\mathcal{G}_t^n, t \geq 0) := (\mathcal{F}_{nt}^n, t \geq 0)$, the associated filtration,

$$\bar{\tau}_0^n := \inf\left\{t \geq 0, \bar{\nu}_t^n(\mathbb{R}_+^*) = 0\right\} = \frac{1}{n}\tau_0^n,$$

$$\bar{\omega}_0^n := \inf\left\{t \geq 0, t_1(\bar{\nu}_t^n) = 0\right\} = \frac{1}{n}\omega_0^n$$

and normalize the arrival process as well as the performance processes of the n-th system the corresponding way, i.e., for all $t \geq 0$,

$$\bar{N}_t^n := \frac{N_{nt}^n}{t},\ \bar{X}_t^n := \frac{X_{nt}^n}{t},\ \bar{Q}_t^n := \frac{Q_{nt}^n}{t},\ \bar{P}_t^n := \frac{P_{nt}^n}{t}.$$

For all $t \geq 0$, $\bar{Q}_t^n$ and $\bar{P}_t^n$ can thus be recovered by:

$$\bar{Q}_t^n = \langle \bar{\nu}_t^n, \mathbf{1}_{\mathbb{R}_+^*} \rangle, \tag{10}$$

$$\bar{P}_t^n = \langle \bar{\nu}_t^n, \mathbf{1}_{\mathbb{R}-} \rangle. \tag{11}$$

Let $\phi \in \mathcal{C}_b^1$ and $\psi^n(.) = \phi\left(./n\right)/n$. As easily seen from (8) and (9), the process defined for all t by

$$\begin{aligned}\bar{M}_\phi^n(t) := M_{\psi^n}^n(nt) = \langle \bar{\nu}_t^n, \phi \rangle - \langle \bar{\nu}_0^n, \phi \rangle - \int_0^t \langle \bar{\nu}_s^n, \phi' \rangle ds \\ + \mu^n \int_0^t \phi\left(t_1(\bar{\nu}_s^n)\right) \mathbf{1}_{\{\bar{\nu}_s^n(\mathbb{R}_+^*)>0\}}\, ds - \lambda^n t \mathbf{E}\left[\phi\left(\frac{D^n}{n}\right)\right]\end{aligned} \tag{12}$$

is a $\mathcal{G}_t^n$-martingale of $\mathcal{D}\left([0,\infty),\mathbb{R}\right)$, such that $\bar{M}_\phi^n(t) \in L^2$ for all t. Its increasing process is given for all t by:

$$< \bar{M}_\phi^n >_t = \frac{\mu^n}{n} \int_0^t \phi^2\left(t_1(\bar{\nu}_s^n)\right) \mathbf{1}_{\{\bar{\nu}_s^n(\mathbb{R}_+^*)>0\}}\, ds + \frac{\lambda^n}{n} t \mathbf{E}\left[\phi^2\left(\frac{D^n}{n}\right)\right].$$

We now define the set of hypothesis under which we will prove a law of large numbers for the sequence of processes ν^n.

Hypothesis 1.

- *There exists two real numbers $\mu > 0$ and $\lambda > \mu$ such that:*

$$\lambda^n \underset{n\to\infty}{\longrightarrow} \lambda, \tag{13}$$

$$\mu^n \underset{n\to\infty}{\longrightarrow} \mu. \tag{14}$$

- *For all $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that for all $n \in \mathbb{N}^*$,*

$$\mathbf{P}\left[\langle \nu_0^n, 1 \rangle > n M_\varepsilon\right] \leq \varepsilon. \tag{15}$$

- *There exists a measure $\bar{\nu}_0^*$ of $\mathcal{M}_f^+$ such that for all $f \in \mathcal{D}_b$:*

$$\left\{\langle \bar{\nu}_0^n, f \rangle\right\}_{n\in\mathbb{N}^*} \xrightarrow{\mathcal{P}} \langle \bar{\nu}_0^*, f \rangle.$$

- *There exist an integrable and almost surely non-negative r.v. $\bar{D}$ such that:*

$$\frac{D^n}{n} \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} \bar{D},$$

$$\mathbf{E}\left[\frac{D^n}{n}\right] \underset{n\to\infty}{\longrightarrow} \mathbf{E}\left[\bar{D}\right].$$

Proposition 1. *Assume that Hypothesis 1 holds. Then, $\left\{(\bar{\nu}_t^n)_{t\geq 0}\right\}_{n\in\mathbb{N}^*}$ is tight in $\mathcal{D}\left([0,\infty),\mathcal{M}_f^+\right)$.*

Proof. According to Jakubowski's criterion [Daw93], it suffices to show that:

1. For all $\phi \in \mathcal{C}_b^1$, the sequence $\left\{(\langle \bar{\nu}_t^n, \phi \rangle)_{t\geq 0}\right\}_{n\in\mathbb{N}^*}$ is tight in $\mathcal{D}\left([0,\infty),\mathbb{R}\right)$,
2. For all $T > 0$ and $0 < \eta < 1$, there exists a compact subset $\mathbf{K}_{T,\eta}$ of $\mathcal{M}_f^+$ such that

$$\liminf_{n\to\infty} \mathbf{P}\left[\bar{\nu}_t^n \in \mathbf{K}_{T,\eta} \forall t \in [0,T]\right] \geq 1 - \eta.$$

In order to prove the first condition, let us fix $\phi \in \mathcal{C}_b^1$ and $T > 0$. Remarking that for all $s \geq 0$,

$$\langle \bar{\nu}_s^n, 1\rangle \leq \bar{N}_s^n + \langle \bar{\nu}_0^n, 1\rangle,$$

Equation (12) yields for all $u < v \leq T$:

$$\begin{aligned}
(16) \quad |\langle \bar{\nu}_v^n, \phi\rangle - \langle \bar{\nu}_u^n, \phi\rangle| &\leq \int_u^v |\langle \bar{\nu}_s^n, \phi'\rangle| \, ds + \mu^n \int_u^v |\phi(t_1(\bar{\nu}_s^n))| \, ds \\
&\qquad + \lambda^n \mathbf{E}\left[\phi\left(\frac{D^n}{n}\right)\right] |v-u| + \left|\bar{M}_\phi^n(v) - \bar{M}_\phi^n(u)\right| \\
&\leq |v-u| \parallel \phi' \parallel_\infty \bar{N}_T^n + |v-u| \parallel \phi' \parallel_\infty \langle \bar{\nu}_0^n, 1\rangle \\
&\quad + |v-u| \parallel \phi \parallel_\infty (|\mu^n - \mu| + |\lambda^n - \lambda|) + |v-u| \parallel \phi \parallel_\infty (\lambda + \mu) \\
&\qquad + \left|\bar{M}_\phi^n(v) - \bar{M}_\phi^n(u)\right|.
\end{aligned}$$

Let $\varepsilon > 0$ and $\eta > 0$. First, let

$$\delta_1 := \frac{\varepsilon\eta}{30 \parallel \phi' \parallel_\infty \lambda T}.$$

From Markov's inequality:

$$\begin{aligned}
\mathbf{P}&\left[\sup_{u,v<T\,,\,|v-u|\leq\delta_1} |v-u| \parallel \phi' \parallel_\infty \bar{N}_T^n \geq \frac{\eta}{5}\right] \\
&\leq \frac{5\delta_1 \parallel \phi' \parallel_\infty}{\eta} \mathbf{E}\left[\bar{N}_T^n\right] \\
&\leq \frac{\lambda T 5\delta_1 \parallel \phi' \parallel_\infty}{\eta} + \frac{|\lambda - \lambda^n| \, T 5\delta_1 \parallel \phi' \parallel_\infty}{\eta} \\
&= \frac{\varepsilon}{6} + \frac{|\lambda - \lambda^n| \, T 5\delta_1 \parallel \phi' \parallel_\infty}{\eta},
\end{aligned}$$

and thus with (13), there exists $N_1 > 0$ such that for all $n \geq N_1$,

$$\mathbf{P}\left[\sup_{u,v<T\,,\,|v-u|\leq\delta_1} |v-u| \parallel \phi' \parallel_\infty \bar{N}_T^n \geq \frac{\eta}{5}\right] \leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.$$

Let now:

$$\delta_2 := \frac{\eta}{5M_{\varepsilon/3} \parallel \phi' \parallel_\infty}.$$

According to (15), for all $n \in \mathbb{N}^*$,

$$\begin{aligned}
\mathbf{P}&\left[\sup_{u,v<T\,,\,|v-u|\leq\delta_2} |v-u| \parallel \phi' \parallel_\infty \langle \bar{\nu}_0^n, 1\rangle \geq \frac{\eta}{5}\right] \\
&\leq \mathbf{P}\left[\langle \bar{\nu}_0^n, 1\rangle \geq \frac{\eta}{5\delta_2 \parallel \phi' \parallel_\infty}\right] = \mathbf{P}\left[\langle \bar{\nu}_0^n, 1\rangle \geq M_{\varepsilon/3}\right] \leq \frac{\varepsilon}{3}.
\end{aligned}$$

According to assumptions (13) and (14), there exists N_2, such that for all $n \geq N_2$, for all $u, v \leq T$,

$$|v-u| \parallel \phi \parallel_\infty (|\mu^n - \mu| + |\lambda^n - \lambda|) < \frac{\eta}{5},$$

and letting

$$\delta_3 := \frac{\eta}{6 \parallel \phi \parallel_\infty (\lambda + \mu)},$$

$$\sup_{u,v<T\,,\,|v-u|\leq\delta_3}|v-u|\parallel\phi\parallel_\infty(\lambda+\mu)\leq\delta_3\parallel\phi\parallel_\infty(\lambda+\mu)<\frac{\eta}{5}.$$

Now, let $\xi>0$. Apply successively Markov's and Doob's inequalities:

$$\begin{aligned}\mathbf{P}\left[\sup_{t\leq T}\left|\bar{M}^n_\phi(t)\right|\geq\xi\right]&\leq\frac{4}{\xi^2}\mathbf{E}\left[<\bar{M}^n_\phi>_T\right]\\&=\frac{4}{\xi^2}\mathbf{E}\left[\frac{\mu^n}{n}\int_0^T\phi^2\left(\mathcal{R}(\bar{\nu}^n_s)\right)\mathbf{1}_{\left\{\bar{\nu}^n_s(\mathbb{R}^*_+)>0\right\}}ds+\frac{\lambda^n}{n}T\mathbf{E}\left[\phi^2\left(\frac{D^n}{n}\right)\right]\right]\\&\leq\frac{4}{\xi^2}\left(\frac{\mu^n}{n}+\frac{\lambda^n}{n}\right)||\,\phi^2\,||_\infty\,T\underset{n\to\infty}{\longrightarrow}0.\end{aligned}$$

For all $n\in\mathbb{N}^*$, $\left(\bar{M}^n_\phi(t),t\geq0\right)$ being a rcll process on $[0,T]$, one can apply the standard convergence criterion [Rob00], from which it follows that $\left\{\left(\bar{M}^n_\phi(t),t\geq0\right)\right\}_{n\in\mathbb{N}^*}$ converges in distribution to the null process. This sequence is in particular tight in $\mathcal{D}([0,T],\mathbb{R})$: there exists $\delta_4>0$ and $N_3>0$ such that for all $n\geq N_3$:

$$\mathbf{P}\left[\sup_{u,v\leq T\,,\,|v-u|\leq\delta_4}\left|\bar{M}^n_\phi(v)-\bar{M}^n_\phi(u)\right|\geq\frac{\eta}{5}\right]\leq\frac{\varepsilon}{3}.$$

Finally, in view of the previous inequalities and (16), there exists $\delta>0$ and $N\in\mathbb{N}$ such that for all $n\geq N$:

$$\mathbf{P}\left[\sup_{u,v\leq T\,,\,|v-u|\leq\delta}\left|\langle\bar{\nu}^n_v,\phi\rangle-\langle\bar{\nu}^n_u,\phi\rangle\right|\geq\eta\right]\leq\varepsilon. \tag{17}$$

On the other hand, let

$$\alpha_\varepsilon:=M_\varepsilon\parallel\phi\parallel_\infty.$$

Assumption (15) implies that for all $n\in\mathbb{N}^*$,

$$\mathbf{P}\left[|\langle\bar{\nu}^n_0,\phi\rangle|>\alpha_\varepsilon\right]\leq\mathbf{P}\Big[\parallel\phi\parallel_\infty\langle\bar{\nu}^n_0,1\rangle>M_\varepsilon\parallel\phi\parallel_\infty\Big]\leq\varepsilon. \tag{18}$$

With (18) and (17) we can apply the standard tightness criterion of real valued processes (see for instance [Rob00]): for all $T>0$, $\left\{(\langle\bar{\nu}^n_t,\phi\rangle)_{t\geq0}\right\}_{n\in\mathbb{N}^*}$ is tight in $\mathcal{D}([0,T],\mathbb{R})$: it is tight in $\mathcal{D}([0,\infty),\mathbb{R})$.

We now prove the second tightness condition (compact containment). Let us first apply [GPW01], Lemma A.2.: under hypothesis 1, we have the following weak law of large numbers:

$$\left\{\left(\frac{1}{n}\sum_{i=1}^{\bar{N}^n_t}\phi(\bar{D}^n_i)\right)_{t\geq0}\right\}_{n\in\mathbb{N}^*}\Longrightarrow\left(\lambda t\mathbf{E}\left[\phi(\bar{D})\right]\right)_{t\geq0}\ \text{in}\ \mathcal{D}([0,T],\mathbb{R}),$$

for any $\phi\in\mathcal{C}_b$. In particular, this yields for any $0<l\leq T$:

$$\mathbf{P}\left[\sup_{t\in[0,T-l]}\frac{1}{n}\sum_{N^n_{nt}+1}^{N^n_{n(t+l)}}\phi(\bar{D}^n_i)>2\lambda l\mathbf{E}\left[\phi(\bar{D})\right]\right]\underset{n\to\infty}{\longrightarrow}0. \tag{19}$$

Taking $l=T$ and $\phi=1$ in the last expression yields:

$$\mathbf{P}\left[\bar{N}^n_T>2\lambda T\right]\underset{n\to\infty}{\longrightarrow}0.$$

Denote $I(.)$, the identity on $\mathbb{R}$. Taking $l = T$ and $\phi = I$ in (19) also leads to:

$$\mathbf{P}\left[\frac{1}{n}\sum_{i=1}^{N^n_{nT}} \bar{D}^n_i > 2\lambda T \mathbf{E}\left[\bar{D}\right]\right] \underset{n\to\infty}{\longrightarrow} 0.$$

Let

$$M_T = \max\left\{2\lambda T + \langle \bar{\nu}^*_0, 1\rangle, 2\lambda T\mathbf{E}\left[\bar{D}\right] + \langle \bar{\nu}^*_0, I\rangle\right\} + 1.$$

We have:

$$\begin{aligned}
(20)\quad & \mathbf{P}\left[\sup_{t\in[0,T]} \max\left\{\langle \bar{\nu}^n_t, \mathbf{1}_{\mathbb{R}+}\rangle, \langle \bar{\nu}^n_t, I\mathbf{1}_{\mathbb{R}+}\rangle\right\} > M_T\right] \\
& \leq \mathbf{P}\left[\langle \bar{\nu}^n_0, 1\rangle + \bar{N}^n_T > 2\lambda T + \langle \bar{\nu}^*_0, 1\rangle + 1\right] \\
& \qquad + \mathbf{P}\left[\langle \bar{\nu}^n_0, I\rangle + \frac{1}{n}\sum_{i=1}^{N^n_{nT}} \bar{D}^n_i > 2\lambda T\mathbf{E}\left[\bar{D}\right] + \langle \bar{\nu}^*_0, I\rangle + 1\right] \\
& \leq \mathbf{P}\left[\langle \bar{\nu}^n_0, 1\rangle > \langle \bar{\nu}^*_0, 1\rangle + 1\right] + \mathbf{P}\left[\bar{N}^n_T > 2\lambda T\right] \\
& \qquad + \mathbf{P}\left[\langle \bar{\nu}^n_0, I\rangle > \langle \bar{\nu}^*_0, I\rangle + 1\right] + \mathbf{P}\left[\frac{1}{n}\sum_{i=1}^{N^n_{nT}} \bar{D}^n_i > 2\lambda T\mathbf{E}\left[\bar{D}\right]\right] \\
& \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Let us now define, for all $T > 0$, $0 < \eta < 1$, the set

$$\mathcal{K}_{T,\eta} := \left\{\zeta \in \mathcal{M}^+_f\,;\, \max\left\{\langle \zeta, \mathbf{1}_{\mathbb{R}+}\rangle, \langle \zeta, I\mathbf{1}_{\mathbb{R}+}\rangle\right\} \leq M_T\,,\, \langle \zeta, \mathbf{1}_{(-\infty,-T]}\rangle = 0\right\}.$$

Since $\langle \zeta, I\mathbf{1}_{\mathbb{R}+}\rangle \leq M_T$, this implies that for all $y > 0$, $\zeta\left([y,\infty)\right) \leq M_T/y$, and thus

$$\lim_{y\to\infty} \sup_{\zeta\in\mathcal{K}_{T,\eta}} \zeta\left([y,\infty)\right) = 0\ ,\ \lim_{y\to-\infty} \sup_{\zeta\in\mathcal{K}_{T,\eta}} \zeta\left((-\infty,y]\right) = 0,$$

which implies that $\mathcal{K}_{T\eta} \subset \mathcal{M}^+_f$ is relatively compact [Kal83]. Now, since up to time T no lost customer can have a residual time credit less than $-T$,

$$\sup_{t\leq T} \langle \bar{\nu}^n_t, \mathbf{1}_{(-\infty,-T]}\rangle = 0.$$

This, together with (20) implies that:

$$\liminf_{n\to\infty} \mathbf{P}\left[\bar{\nu}^n_t \in \mathcal{K}_{T,\eta},\ \text{for all } t\in[0,T]\right] \geq 1-\eta.$$

$\mathbf{K}_{T,\eta}$ being the closure of $\mathcal{K}_{T,\eta}$, we found a compact subset $\mathbf{K}_{T,\eta} \subset \mathcal{M}^+_f$ such that:

$$\liminf_{n\to\infty} \mathbf{P}\left[\bar{\nu}^n_t \in \mathbf{K}_{T,\eta},\ \text{for all } t\in[0,T]\right] \geq 1-\eta.$$

□

Theorem 3 (Fluid limit theorem for M/M/1/1+GI-EDF queues)**.** *Assume that Hypothesis 1 holds, that there exists $T > 0$ such that:*

$$\mathbf{P}\left[\bar{\tau}^n_0 \leq T\right] \underset{n\to\infty}{\longrightarrow} 0, \tag{21}$$

and that there exists a deterministic element $(\bar{r}_t)_{t\geq 0}$ *of* $\mathcal{C}([0,\infty),\mathbb{R})$ *such that:*

$$(t_1(\bar{\nu}^n_t))_{t\geq 0} \Longrightarrow (\bar{r}_t)_{t\geq 0} \;\; \text{in } \mathcal{D}([0,T],\mathbb{R}). \tag{22}$$

Then:

$$\left\{(\bar{\nu}^n_t)_{t\geq 0}\right\}_{n\in\mathbb{N}^*} \Longrightarrow (\bar{\nu}^*_t)_{t\geq 0} \;\; \text{in } \mathcal{D}\left([0,T],\mathcal{M}^+_f\right),$$

where $(\bar{\nu}^*_t)_{t\geq 0}$ *is the deterministic element of* $\mathcal{C}\left([0,T],\mathcal{M}^+_f\right)$ *defined for all* $f\in\mathcal{D}_b$ *and all* $t\in[0,T]$ *by:*

$$\langle \bar{\nu}^*_t, f\rangle = \langle \bar{\nu}^*_0, \tau_t\, f\rangle - \mu\int_0^t \tau_{t-s} f\left(\bar{r}_s\right)\, ds + \lambda\int_0^t \mathbf{E}\left[\tau_{t-s} f\left(\bar{D}\right)\right]\, ds.$$

Proof. Let $(\bar{\chi}_t)_{t\geq 0}$ be a limit point of $\left\{(\bar{\nu}^n_t)_{t\geq 0}\right\}_{n\in\mathbb{N}^*}$. On the one hand, assumption (21) implies that $\left\{\mathbf{1}_{\{\bar{\tau}^n_0\leq T\}}\right\}_{n\in\mathbb{N}^*}$ converges in distribution to 0. On another hand, for all $\phi\in\mathcal{C}^1_b$ the mappings

$$\Psi_1 : \begin{cases} \mathcal{D}([0,\infty),\mathbb{R}) & \hookrightarrow \mathcal{D}([0,\infty),\mathbb{R}) \\ (X_t)_{t\geq 0} & \mapsto (\phi(X_t))_{t\geq 0} \end{cases}$$

and

$$\Psi_2 : \begin{cases} \mathcal{D}([0,\infty),\mathbb{R}) & \hookrightarrow \mathcal{C}([0,\infty),\mathbb{R}) \\ (Y_t)_{t\geq 0} & \mapsto \left(\int_0^t Y_s\, ds\right)_{t\geq 0} \end{cases}$$

are continuous, as well as $\Psi := \Psi_2\circ\Psi_1$, hence in view of (22), the continuous mapping theorem entails that:

$$\left\{\left(\int_0^t \phi\left(t_1(\bar{\nu}^n_s)\right)\, ds\right)_{t\geq 0}\right\}_{n\in\mathbb{N}^*} \Longrightarrow \left(\int_0^t \phi(\bar{r}_s)\, ds\right)_{t\geq 0} \;\; \text{in } \mathcal{D}([0,T],\mathbb{R}).$$

Consequently, for all $t\in[0,T]$, all $\phi\in\mathcal{C}^1_b$:

$$\langle \bar{\chi}_t, \phi\rangle = \langle \bar{\nu}^*_0, \phi\rangle - \int_0^t \langle \bar{\chi}_s, \phi'\rangle\, ds - \mu\int_0^t \phi\left(\bar{r}_s\right)\, ds + \lambda t\mathbf{E}\left[\phi(\bar{D})\right]. \tag{23}$$

In particular, the latter is true for all $\phi\in\mathcal{S}$: (23) is the integrated transport equation $(\mathrm{E}(\bar{\nu}^*_0, g, 1))$, where g is defined by: for all $\phi\in\mathcal{S}$,

$$\langle g_t, \phi\rangle := -\mu\int_0^t \phi\left(\bar{r}_s\right)\, ds + \lambda t\mathbf{E}\left[\phi(\bar{D})\right].$$

According to Theorem 1, the only solution of (23) is given for all $t\in[0,T]$ and all $\phi\in\mathcal{S}$ by:

$$\begin{aligned}
\langle \bar{\chi}_t, \phi\rangle &= \langle \bar{\nu}^*_0, \tau_t\,\phi\rangle + \langle g_t, \phi\rangle - \int_0^t \langle g_s, \tau_{t-s}\,\phi'\rangle\, ds \\
&= \langle \bar{\nu}^*_0, \tau_t\,\phi\rangle - \mu\int_0^t \phi\left(\bar{r}_s\right)\, ds + \lambda t\mathbf{E}\left[\phi(\bar{D})\right] \\
&\qquad + \mu\int_0^t\int_0^s \left(\tau_{t-s}\phi'\left(\bar{r}_u\right)\right)\, du\, ds - \lambda\int_0^t s\mathbf{E}\left[\tau_{(t-s)}\phi'\left(\bar{D}\right)\right]\, ds \\
&= \langle \bar{\nu}^*_t, \phi\rangle.
\end{aligned}$$

The limit point is therefore unique in $\mathcal{D}\left([0,T],\mathcal{M}_f^+\right)$, equal to $(\bar{\nu}_t^*)_{t\geq 0}$, since $\mathcal{S}$ is a separating class of $\mathcal{M}_f^+$. □

7. Applications

M/M/1/1+D-EDF case. We hereafter apply Theorem 3 to determine the fluid limit of the M/M/1/1+GI-EDF system in which the time credits of the customers are deterministic. We verify in particular that assumptions (21) and (22) are satisfied in this case, and specify the form of the limit.

We therefore consider a sequence of M/M/1/1+D-EDF systems, for which we make the following assumptions:

Hypothesis 2 (Basic Assumptions for a M/M/1/1+D-EDF system)**.**

- *For all $n\in\mathbb{N}^*$, there are initially $n+1$ customers in the buffer, all of them with time credit nd, where $d>0$ (that is, the n customers have their deadline at time nd),*
- *for all $n\in\mathbb{N}^*$, λ^n is the intensity of the Poisson process of arrivals, where*
$$\lambda^n \underset{n\to\infty}{\longrightarrow} \lambda>0,$$
- *for all $n\in\mathbb{N}^*$ the customers require service durations exponentially distributed, of parameter μ^n, satisfying:*
$$\mu^n \underset{n\to\infty}{\longrightarrow} \mu\,,\ \text{ where } (d)^{-1}<\mu<\lambda,$$
- *for all $n\in\mathbb{N}^*$, the initial time credit of any customer is deterministic, given by d^n, where $d^n/n \underset{n\to\infty}{\longrightarrow} d$.*

This queueing system is described by the profile process $\left(\bar{\nu}_t^{n,\mathrm{D}}\right)_{t\geq 0}$, associated to $(n\delta_{nd},\lambda^n,\mu^n,d^n)$, which keeps track of all the residual time credits of all the customers waiting in the buffer are already lost. The notations are those of the preceding sections, with superscripts $^{\mathrm{D}}$, for "deterministic".

Lemma 1. *For all $x<\mu^{-1}$,*
$$\mathbf{P}\left[\bar{\tau}_0^{n,D}\leq x\right] \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. For the event
$$\left\{\bar{\tau}_0^{n,\mathrm{D}}\leq x\right\}=\left\{\tau_0^{n,\mathrm{D}}\leq nx\right\}$$
to occur, the $n+1$ customers initially present in the buffer must have all entered the service before time nx, since they couldn't have been eliminated before $nx\leq n/\mu<nd$. Therefore the first n customers among them must have completed their service before nx, or in other words:
$$\mathbf{P}\left[\bar{\tau}_0^{n,\mathrm{D}}\leq x\right]=\mathbf{P}\left[\tau_0^{n,\mathrm{D}}\leq nx\right]\leq \mathbf{P}\left[S_{nx}^n\geq n\right],$$
where $(S_t^n)_{t\geq 0}$ denotes a Poisson process of intensity μ^n (the server works without interruption at least until he has completed the services of these customers). Hence, S_{nx}^n has the same distribution as the sum of n independent r.v $(P_i^n, i=1,...,n)$, Poisson distributed of parameter $\mu^n x$. Hence, denoting $(P_i, i=1,...,n)$, a family of n independent r.v. Poisson distributed of parameter μ,
$$\mathbf{P}\left[\bar{\tau}_0^{n,\mathrm{D}}\leq x\right]\ \leq\ \mathbf{P}\left[\frac{1}{n}\sum_{i=1}^n P_i^n\geq 1\right]\ \underset{\infty}{\sim}\ \mathbf{P}\left[\frac{1}{n}\sum_{i=1}^n P_i\geq 1\right]\underset{n\to\infty}{\longrightarrow}\mathbf{1}_{\{x\mu\geq 1\}}\ =\ 0,$$

according to the weak law of large numbers. □

Lemma 2. *For all $\xi > 0$,*

$$\mathbf{P}\left[\bar{\omega}_0^{n,D} \le \bar{\omega}_0^{*,D} - \xi\right] \underset{n\to\infty}{\longrightarrow} 0,$$

where

$$\bar{\omega}_0^{*,D} := \frac{\rho d - \mu^{-1}}{\rho - 1}.$$

Proof. We have:

$$\begin{aligned}(24)\quad \mathbf{P}\left[\bar{\omega}_0^{n,\mathrm{D}} \le \bar{\omega}_0^{*,\mathrm{D}} - \xi\right] &= \mathbf{P}\left[\left\{\omega_0^{n,\mathrm{D}} \le n\bar{\omega}_0^{*,\mathrm{D}} - n\xi\right\} \cap \left\{n\bar{\omega}_0^{*,\mathrm{D}} - n\xi \le \tau_0^{n,\mathrm{D}}\right\}\right] \\ &\quad + \mathbf{P}\left[\left\{\omega_0^{n,\mathrm{D}} \le \tau_0^{n,\mathrm{D}}\right\} \cap \left\{n\bar{\omega}_0^{*,\mathrm{D}} - n\xi > \tau_0^{n,\mathrm{D}}\right\}\right] \\ &\quad + \mathbf{P}\left[\left\{\tau_0^{n,\mathrm{D}} \le \omega_0^{n,\mathrm{D}} \le n\bar{\omega}_0^{*,\mathrm{D}} - n\xi\right\} \cap \left\{n\bar{\omega}_0^{*,\mathrm{D}} - n\xi > \tau_0^{n,\mathrm{D}}\right\}\right] \\ &\le \mathbf{P}\left[\omega_0^{n,\mathrm{D}} \le (n\bar{\omega}_0^{*,\mathrm{D}} - n\xi) \wedge \tau_0^{n,\mathrm{D}}\right] + \mathbf{P}\left[\tau_0^{n,\mathrm{D}} \le \omega_0^{n,\mathrm{D}} \le n\bar{\omega}_0^{*,\mathrm{D}} - n\xi\right].\end{aligned}$$

Let us denote for all $t \ge 0$:

$$\begin{cases} \mathcal{N}_t^n := & \text{number of customers arrived up to time } t, \\ & \quad \text{having deadline before } t \\ \mathcal{S}_t^n := & \text{number of services completed up to } t. \end{cases}$$

On the event

$$\left\{\omega_0^{n,\mathrm{D}} \le (n\bar{\omega}_0^{*,\mathrm{D}} - n\xi) \wedge \tau_0^{n,\mathrm{D}}\right\},$$

there is at least one loss and no idle time before time $(n\bar{\omega}_0^{*,\mathrm{D}} - n\xi) \wedge \tau_0^{n,\mathrm{D}}$. For this to occur, since the service discipline amounts to FIFO, there must be at the first time of loss, say $nt \ge nd$, the number of customers initially in the system or arrived up to nt, of a priority higher to the priority of the customer who is lost at nt (there are $n + 1 + \mathcal{N}_{nt}^n$ such customers) must be greater than the number of services initiated up to nt (i.e., $\mathcal{S}_{nt}^n + 1$). We can therefore write that:

$$\begin{aligned}(25)\quad & \mathbf{P}\left[\omega_0^{n,\mathrm{D}} \le (n\bar{\omega}_0^{*,\mathrm{D}} - n\xi) \wedge \tau_0^{n,\mathrm{D}}\right] \\ &\le \mathbf{P}\left[\sup_{d \le t \le (\bar{\omega}_0^{*,\mathrm{D}} - \xi) \wedge \bar{\tau}_0^{n,\mathrm{D}}} (\mathcal{N}_{nt}^n + n + 1) - (\mathcal{S}_{nt}^n + 1) \ge 0\right] \\ &= \mathbf{P}\left[\sup_{d \le t \le (\bar{\omega}_0^{*,\mathrm{D}} - \xi) \wedge \bar{\tau}_0^{n,\mathrm{D}}} \mathcal{N}_{nt}^n - \mathcal{S}_{nt}^n \ge -n\right].\end{aligned}$$

Since there has been no idle time in $\left[0, n(\bar{\omega}_0^{*,\mathrm{D}} - \xi) \wedge \tau_0^{n,\mathrm{D}}\right]$, for all t in this interval, $\mathcal{S}_{nt}^n$ has the same distribution as S_{nt}^n, where $(S_t^n)_{t\ge 0}$ is a Poisson process of intensity μ^n. On the other hand, the process of arrivals $(N_t^n)_{t\ge 0}$ marked by the initial time credits of the customers $\{D_i^n\}_{i\in\mathbb{N}^*}$ being a two-dimensional Poisson process, it is easily checked that the process $(\mathcal{N}_t^n - \lambda_t^n)_{t\ge 0}$ is an $\mathcal{F}_t^n$-martingale, where

$$\lambda_t^n = \lambda^n \left(t - nd\right)^+.$$

Thus the process defined for all t by

$$\mathcal{M}_t^n := \mathcal{N}_{nt}^n - S_{nt}^n - (\lambda_{nt}^n - \mu^n nt)$$

is a $\mathcal{G}_t^n$-martingale. Hence, with (25):

$$\begin{aligned}
(26)\quad & \mathbf{P}\left[\omega_0^{n,\mathrm{D}} \leq (n\bar{\omega}_0^{*,\mathrm{D}} - n\xi) \wedge \tau_0^{n,\mathrm{D}}\right] \\
& \leq \mathbf{P}\left[\sup_{d\leq t\leq \bar{\omega}_0^{*,\mathrm{D}}-\xi} \mathcal{M}_t^n \geq \inf_{d\leq t\leq \bar{\omega}_0^{*,\mathrm{D}}-\xi} (\mu^n nt - \lambda_{nt}^n) - n\right] \\
& = \mathbf{P}\left[\sup_{t\leq \bar{\omega}_0^{*,\mathrm{D}}-\xi} \mathcal{M}_t^n \geq n\left\{\lambda^n d - 1 - (\lambda^n - \mu^n)\left(\bar{\omega}_0^{*,\mathrm{D}} - \xi\right)\right\}\right] \\
& \leq \frac{4}{n^2\left\{\lambda^n d - 1 - (\lambda^n - \mu^n)\left(\bar{\omega}_0^{*,\mathrm{D}} - \xi\right)\right\}^2}\mathbf{E}\left[\langle \mathcal{M}^n\rangle_{\bar{\omega}_0^{*,\mathrm{D}}-\xi}\right] \\
& \underset{n\to\infty}{\sim} \frac{4}{(n(\lambda-\mu)\xi)^2}\left\{n\lambda^n\left(\bar{\omega}_0^{*,\mathrm{D}} - \xi - d\right) + \mu^n n\left(\bar{\omega}_0^{*,\mathrm{D}} - \xi\right)\right\} \\
& \qquad \underset{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

using successively Doob's inequality and the fact that $(\lambda-\mu)\bar{\omega}_0^{*,\mathrm{D}} = \lambda d - 1$. Now, clearly

$$\begin{aligned}
(27)\quad & \mathbf{P}\left[\tau_0^{n,\mathrm{D}} \leq \omega_0^{n,\mathrm{D}} \leq n\bar{\omega}_0^{*,\mathrm{D}} - n\xi\right] \\
& \leq \mathbf{P}\left[\frac{n}{2\mu} \leq \tau_0^{n,\mathrm{D}} \leq \omega_0^{n,\mathrm{D}} \leq n\bar{\omega}_0^{*,\mathrm{D}} - n\xi\right] + \mathbf{P}\left[\tau_0^{n,\mathrm{D}} \leq \frac{n}{2\mu}\right].
\end{aligned}$$

on the event

$$\left\{\frac{n}{2\mu} \leq \tau_0^{n,\mathrm{D}} \leq \omega_0^{n,\mathrm{D}} \leq n\bar{\omega}_0^{*,\mathrm{D}} - n\xi\right\},$$

there exists $t \in \left[\frac{1}{2\mu}, \bar{\omega}_0^{*,\mathrm{D}} - \xi\right]$ (the first one) such that the buffer is empty at nt, but there has been no loss before nt. For this event to occur, there must be up to nt, the same number of customers entered (i.e., N_{nt}^n) as of services initiated (i.e., $\mathcal{S}_{nt}^n + 1$). Thus, remarking that the process defined for all t by:

$$M_t^n := N_{nt}^n - S_{nt}^n - (\lambda^n nt - \mu^n nt)$$

is a $\mathcal{G}_t^n$-martingale, and that for all $t \leq \tau_0^{n,\mathrm{D}}$, $\mathcal{S}_{nt}^n$ equals S_{nt}^n in distribution,

$$\begin{aligned}
& \mathbf{P}\left[\frac{n}{2\mu} \leq \tau_0^{n,\mathrm{D}} \leq \omega_0^{n,\mathrm{D}} \leq n\bar{\omega}_0^{*,\mathrm{D}} - n\xi\right] \\
& \leq \mathbf{P}\left[\sup_{1/(2\mu)\leq t\leq \bar{\omega}_0^{*,\mathrm{D}}-\xi} \mathcal{S}_{nt}^n - N_{nt}^n \geq -1\right] \\
& \leq \mathbf{P}\left[\sup_{1/(2\mu)\leq t\leq \bar{\omega}_0^{*,\mathrm{D}}-\xi} - M_t^n \geq -1 + n(\lambda^n - \mu^n)\frac{1}{2\mu}\right] \\
& \leq \frac{4}{\left(n(\lambda^n-\mu^n)\frac{1}{2\mu} - 1\right)^2}\mathbf{E}\left[< M^n >_{\bar{\omega}_0^{*,\mathrm{D}}-\xi}\right] \\
& \underset{n\to\infty}{\sim} \frac{16}{(n(\rho-1)-2)^2}\left\{(\lambda^n+\mu^n)\, n(\bar{\omega}_0^{*,\mathrm{D}} - \xi)\right\} \underset{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

which, together with Lemma 1 applied to $x = 1/(2\mu)$ and (27) yields:

$$\mathbf{P}\left[\tau_0^{n,\mathrm{D}} \leq \omega_0^{n,\mathrm{D}} \leq n\bar{\omega}_0^{*,\mathrm{D}} - n\xi\right] \underset{n\to\infty}{\longrightarrow} 0. \tag{28}$$

We conclude by substituting (26) and (28) in (24). □

Proposition 2. *Assume Hypothesis 1 and 2 holds, then*

$$\left\{\left(t_1(\bar{\nu}_t^{n,D})\right)_{t\geq 0}\right\}_{n\in\mathbb{N}^*} \Longrightarrow \left(\bar{r}_t^D\right)_{t\geq 0} \text{ in } \mathcal{D}\left([0,\infty),\mathbb{R}\right),$$

where $(\bar{r}_t^D)_{t\geq 0}$ *is the deterministic element of* $\mathcal{C}\left([0,\infty),\mathbb{R}\right)$ *defined for all* t *by:*

$$\bar{r}_t^D = \left\{d - t\mathbf{1}_{\{t\leq\mu^{-1}\}} - \left(\frac{\rho-1}{\rho}t + \frac{1}{\lambda}\right)\mathbf{1}_{\{t>\mu^{-1}\}}\right\}^+. \tag{29}$$

Proof. Fix $\xi > 0$, assuming without loss of generality that:

$$\xi < \frac{d-\mu^{-1}}{\rho} \wedge \frac{1}{\mu}, \tag{30}$$

which implies that:

$$\mu^{-1} + \rho\xi < d < \bar{\omega}_0^{*,\mathrm{D}} - \xi. \tag{31}$$

If $\rho < 3$, assume in addition to (30) that:

$$\xi < \frac{1}{(4-\rho)\mu}. \tag{32}$$

Let us first focus on the interval of time $[0, \bar{\omega}_0^{*,\mathrm{D}} - \xi]$. We have:

$$\begin{aligned}\mathbf{P}&\left[\sup_{0\leq t\leq \bar{\omega}_0^{*,\mathrm{D}}-\xi}\left|t_1(\bar{\nu}_t^{n,\mathrm{D}}) - \bar{r}_t^{\mathrm{D}}\right| > \xi\right] \\ &\leq \mathbf{P}\left[\sup_{0\leq t\leq\left\{(\bar{\omega}_0^{*,\mathrm{D}}-\xi)\wedge\bar{\omega}_0^{n,\mathrm{D}}\wedge\bar{\tau}_0^{n,\mathrm{D}}\right\}}\left|t_1(\nu_{nt}^{\mathrm{D}}) - n\bar{r}_t^{\mathrm{D}}\right| > n\xi\right] \\ &\qquad + \mathbf{P}\left[\bar{\omega}_0^{*,\mathrm{D}} - \bar{\omega}_0^{n,\mathrm{D}} \geq \xi\right] + \mathbf{P}\left[\tau_0^{n,\mathrm{D}} \leq n\bar{\omega}_0^{*,\mathrm{D}} - n\xi\right].\end{aligned} \tag{33}$$

Let us denote for all $y \in \mathbb{R}$ and $t \geq 0$:

$\tilde{\mathcal{N}}_{t,y}^n :=$ number of customers arrived up to time t, having deadline before $t + y$.

On the one hand, on the event

$$\left\{\sup_{0\leq t\leq\left\{(\bar{\omega}_0^{*,\mathrm{D}}-\xi)\wedge\bar{\omega}_0^{n,\mathrm{D}}\wedge\bar{\tau}_0^{n,\mathrm{D}}\right\}}\left(t_1(\nu_{nt}^{\mathrm{D}}) - n\bar{r}_t^{\mathrm{D}}\right) > n\xi\right\},$$

there exists $t \in \left[0, \left\{(\bar{\omega}_0^{*,\mathrm{D}} - \xi) \wedge \bar{\omega}_0^{n,\mathrm{D}} \wedge \bar{\tau}_0^{n,\mathrm{D}}\right\}\right]$ such that $t_1(\nu_{nt}^{\mathrm{D}}) - n\bar{r}_t^{\mathrm{D}} > n\xi$. Hence, since there has been no loss until nt, the number of services initiated up to nt (i.e., $\mathcal{S}_{nt}^n + 1$, which equals $S_{nt}^n + 1$ in distribution) is larger than the number of customers initially present, or arrived up to time nt, having deadline before $nt + n\bar{r}_t^{\mathrm{D}} + n\xi$

(i.e., $\tilde{\mathcal{N}}^n_{nt,n\left(\bar{r}^{\mathrm{D}}_t+\xi\right)}+n+1$).
On the other hand, on the event

$$\left\{\sup_{0\le t\le\left\{(\bar{\omega}^{*,\mathrm{D}}_0-\xi)\wedge\bar{\omega}^{n,\mathrm{D}}_0\wedge\bar{\tau}^{n,\mathrm{D}}_0\right\}}\left(n\bar{r}^{\mathrm{D}}_t-t_1(\nu^{\mathrm{D}}_{nt})\right)>n\xi\right\}, \tag{34}$$

there exists $t\in\left[0,(\bar{\omega}^{*,\mathrm{D}}_0-\xi)\wedge\bar{\omega}^{n,\mathrm{D}}_0\wedge\bar{\tau}^{n,\mathrm{D}}_0\right]$ such that $t_1(\nu^{\mathrm{D}}_{nt})<n\bar{r}^{\mathrm{D}}_t-n\xi$. But for all $t\le(\mu^{-1})+\rho\xi$, every customer initially in the system, or arrived before nt has a deadline at, or posterior to, nd, and hence a residual time credit at nt larger or equal to $nd-nt\ge n\bar{r}^{\mathrm{D}}_t-n\xi$. Therefore, for the event (34) to occur, there must exist an instant $t\in\left[\mu^{-1}+\rho\xi,(\bar{\omega}^{*,\mathrm{D}}_0-\xi)\wedge\bar{\omega}^{n,\mathrm{D}}_0\wedge\bar{\tau}^{n,\mathrm{D}}_0\right]$ such that $t_1(\nu^{\mathrm{D}}_{nt})<n\bar{r}^{\mathrm{D}}_t-n\xi$. Since there has been no idle time on the interval $[0,\omega^{n,\mathrm{D}}_0\wedge\tau^{n,\mathrm{D}}_0]$, and since the discipline amounts to FIFO, this implies that $\mathcal{S}^n_{nt}+1$ (equal in distribution to $S^n_{nt}+1$) is less than the number of customers arrived up to time nt, having deadline before $nt+n\bar{r}^{\mathrm{D}}_t-n\xi$ (that is, $\tilde{\mathcal{N}}^n_{nt,n\left(\bar{r}^{\mathrm{D}}_t-\xi\right)}+n+1$). Consequently:

$$\begin{aligned}
&\mathbf{P}\left[\sup_{0\le t\le\left\{(\bar{\omega}^{*,\mathrm{D}}_0-\xi)\wedge\bar{\omega}^{n,\mathrm{D}}_0\wedge\bar{\tau}^{n,\mathrm{D}}_0\right\}}\left|t_1(\nu^{\mathrm{D}}_{nt})-n\bar{r}^{\mathrm{D}}_t\right|>n\xi\right]\\
&\le\mathbf{P}\left[\sup_{0\le t\le\left\{(\bar{\omega}^{*,\mathrm{D}}_0-\xi)\wedge\bar{\omega}^{n,\mathrm{D}}_0\wedge\bar{\tau}^{n,\mathrm{D}}_0\right\}}\left(t_1(\nu^{\mathrm{D}}_{nt})-n\bar{r}^{\mathrm{D}}_t\right)>n\xi\right]\\
&\qquad+\mathbf{P}\left[\sup_{0\le t\le\left\{(\bar{\omega}^{*,\mathrm{D}}_0-\xi)\wedge\bar{\omega}^{n,\mathrm{D}}_0\wedge\bar{\tau}^{n,\mathrm{D}}_0\right\}}\left(n\bar{r}^{\mathrm{D}}_t-t_1(\nu^{\mathrm{D}}_{nt})\right)>n\xi\right]\\
&\le\mathbf{P}\left[\sup_{0\le t\le\bar{\omega}^{*,\mathrm{D}}_0-\xi}\left(S^n_{nt}-\tilde{\mathcal{N}}^n_{nt,n(\bar{r}^{\mathrm{D}}_t+\xi)}\right)\ge n\right]\\
&\qquad+\mathbf{P}\left[\sup_{\mu^{-1}+\rho\xi\le t\le\bar{\omega}^{*,\mathrm{D}}_0-\xi}\left(\tilde{\mathcal{N}}^n_{nt,n(\bar{r}^{\mathrm{D}}_t-\xi)}-S^n_{nt}\right)\ge -n\right].
\end{aligned} \tag{35}$$

It is easily checked, that for all n and all $z\in\mathbb{R}$,

$$\left(\tilde{\mathcal{N}}^n_{nt,n(\bar{r}^{\mathrm{D}}_t+z)}-\tilde{\lambda}^n_{t,z}\right)_{t\ge0}$$

is a $\mathcal{G}^n_t$-martingale, where for all $t\ge0$:

$$\tilde{\lambda}^n_{t,z}=n\lambda^n\left\{t-\left(d-(\bar{r}^{\mathrm{D}}_t+z)\right)^+\right\}^+.$$

Thus, the processes

$$\tilde{\mathcal{M}}^n_\xi(t):=S^n_{nt}-\tilde{\mathcal{N}}^n_{nt,n\left(\bar{r}^{\mathrm{D}}_t+\xi\right)}-\left(\mu^n nt-\tilde{\lambda}^n_{t,\xi}\right)$$

and

$$\widehat{\mathcal{M}}^n_\xi(t):=\tilde{\mathcal{N}}^n_{nt,n\left(\bar{r}^{\mathrm{D}}_t-\xi\right)}-S^n_{nt}-\left(\tilde{\lambda}^n_{t,-\xi}-\mu^n nt\right)$$

are $\mathcal{G}^n_t-$ martingales. Hence, (35) becomes:

$$\begin{aligned}
(36)\quad & \mathbf{P}\left[\sup_{0\le t\le\left\{(\bar\omega_0^{*,\mathrm{D}}-\xi)\wedge\bar\omega_0^{n,\mathrm{D}}\wedge\bar\tau_0^{n,\mathrm{D}}\right\}}\left|t_1(\nu^{\mathrm{D}}_{nt})-n\bar r^{\mathrm{D}}_t\right|>n\xi\right]\\
&\le \mathbf{P}\left[\sup_{0\le t\le\bar\omega_0^{*,\mathrm{D}}-\xi}\tilde{\mathcal{M}}^n_\xi(t)\ge n+\inf_{0\le t\le\bar\omega_0^{*,\mathrm{D}}-\xi}\left(\tilde\lambda^n_{t,\xi}-\mu^n nt\right)\right]\\
&\qquad +\mathbf{P}\left[\sup_{0\le t\le\bar\omega_0^{*,\mathrm{D}}-\xi}\widehat{\mathcal{M}}^n_\xi(t)\ge -n+\inf_{\mu^{-1}+\rho\xi\le t\le\bar\omega_0^{*,\mathrm{D}}-\xi}\left(\mu^n nt-\tilde\lambda^n_{t,-\xi}\right)\right].
\end{aligned}$$

On the one hand, since:

$$\tilde\lambda^n_{t,\xi}-\mu^n nt=\begin{cases} n\left\{\lambda^n(t\wedge\xi)-\mu^n t\right\} & \text{if } t\in\left[0,\mu^{-1}\right]\\ n\left\{\frac{\lambda^n\mu}{\lambda}t-\frac{\lambda^n}{\lambda}+\lambda^n\xi-\mu^n t\right\} & \text{if } t\in\left[\mu^{-1},\bar\omega_0^{*,\mathrm{D}}-\xi\right],\end{cases}$$

it follows from Hypothesis 2 that for a sufficiently large n,

$$n+\inf_{0\le t\le\bar\omega_0^{*,\mathrm{D}}-\xi}\left(\tilde\lambda^n_{t,\xi}-\mu^n nt\right)\ge n+n\left\{\frac{\lambda\xi}{2}-1\right\}>0.$$

On the other hand, since for all $t\in\left[\mu^{-1}+\rho\xi,\bar\omega_0^{*,\mathrm{D}}-\xi\right]$,

$$\mu^n nt-\tilde\lambda^n_{t,-\xi}=n\left\{\mu^n t-\frac{\lambda^n\mu}{\lambda}t+\frac{\lambda^n}{\lambda}+\lambda^n\xi\right\},$$

for a sufficiently large n,

$$-n+\inf_{\mu^{-1}+\rho\xi\le t\le\bar\omega_0^{*,\mathrm{D}}-\xi}\left(\mu^n nt-\tilde\lambda^n_{t,-\xi}\right)\ge -n+n\left\{\frac{\lambda\xi}{2}+1\right\}>0.$$

Thus from (36), for a sufficiently large n,

$$\begin{aligned}
(37)\quad & \mathbf{P}\left[\sup_{0\le t\le\left\{(\bar\omega_0^{*,\mathrm{D}}-\xi)\wedge\bar\omega_0^{n,\mathrm{D}}\wedge\bar\tau_0^{n,\mathrm{D}}\right\}}\left|t_1(\nu^{\mathrm{D}}_{nt})-n\bar r^{\mathrm{D}}_t\right|>n\xi\right]\\
&\le\frac{16}{n\lambda\xi^2}\left\{\mathbf{E}\left[\left\{\tilde{\mathcal{M}}^n_\xi\left(\bar\omega_0^{*,\mathrm{D}}-\xi\right)\right\}^2\right]+\mathbf{E}\left[\left\{\widehat{\mathcal{M}}^n_\xi\left(\bar\omega_0^{*,\mathrm{D}}-\xi\right)\right\}^2\right]\right\}\\
&\qquad=\frac{16}{(n\lambda\xi)^2}\left\{\tilde\lambda^n_{(\bar\omega_0^{*,\mathrm{D}}-\xi),\xi}+\tilde\lambda^n_{(\bar\omega_0^{*,\mathrm{D}}-\xi),-\xi}+2\mu^n n(\bar\omega_0^{*,\mathrm{D}}-\xi)\right\}\xrightarrow[n\to\infty]{}0,
\end{aligned}$$

using successively Tchebitchef and Doob's inequalities.

Consider now the term:

$$\begin{aligned}
(38)\quad \mathbf{P}\left[\tau_0^{n,\mathrm{D}}\le n\bar\omega_0^{*,\mathrm{D}}-n\xi\right] &\le \mathbf{P}\left[\omega_0^{n,\mathrm{D}}\le n\bar\omega_0^{*,\mathrm{D}}-n\xi\right]\\
&\quad+\mathbf{P}\left[\left\{\tau_0^{n,\mathrm{D}}\le n\bar\omega_0^{*,\mathrm{D}}-n\xi\right\}\cap\left\{\omega_0^{n,\mathrm{D}}>n\bar\omega_0^{*,\mathrm{D}}-n\xi\right\}\right].
\end{aligned}$$

On the event

$$\left\{\tau_0^{n,\mathrm{D}}\le n\bar\omega_0^{*,\mathrm{D}}-n\xi\right\}\cap\left\{\omega_0^{n,\mathrm{D}}>n\bar\omega_0^{*,\mathrm{D}}-n\xi\right\},$$

there exists an instant, say $t \in [0, \bar{\omega}_0^{*,\mathrm{D}} - \xi]$ such that the buffer is empty at nt, but there has been no loss before nt. Applying the same arguments that led to (28) yields:

$$\mathbf{P}\left[\left\{\tau_0^{n,\mathrm{D}} \leq n\bar{\omega}_0^{*,\mathrm{D}} - n\xi\right\} \cap \left\{\omega_0^{n,\mathrm{D}} > n\bar{\omega}_0^{*,\mathrm{D}} - n\xi\right\}\right] \underset{n\to\infty}{\longrightarrow} 0,$$

which together with Lemma 2 and (38) yields:

$$\mathbf{P}\left[\tau_0^{n,\mathrm{D}} \leq n\bar{\omega}_0^{*,\mathrm{D}} - n\xi\right] \underset{n\to\infty}{\longrightarrow} 0. \tag{39}$$

Finally, using (37) together with (39) and Lemma 2 in (33):

$$\mathbf{P}\left[\sup_{0\leq t\leq \bar{\omega}_0^{*,\mathrm{D}}-\xi} \left|t_1(\bar{\nu}_t^{n,\mathrm{D}}) - \bar{r}_t^{\mathrm{D}}\right| > \xi\right] \underset{n\to\infty}{\longrightarrow} 0. \tag{40}$$

Let us now consider the interval of time $[\bar{\omega}_0^{*,\mathrm{D}} - \xi, \infty)$. We have:

$$\begin{aligned}
&\mathbf{P}\left[\sup_{t\geq \bar{\omega}_0^{*,\mathrm{D}}-\xi} \left|t_1(\bar{\nu}_t^{n,\mathrm{D}}) - \bar{r}_t^{\mathrm{D}}\right| > 2\xi\right] \\
&\quad\leq \mathbf{P}\left[\sup_{t\geq \bar{\omega}_0^{*,\mathrm{D}}-\xi} \left(t_1(\nu_{nt}^{\mathrm{D}}) - n\bar{r}_t^{\mathrm{D}}\right) > 2n\xi\right] \\
&\qquad\qquad + \mathbf{P}\left[\inf_{t\geq \bar{\omega}_0^{*,\mathrm{D}}-\xi} t_1(\nu_{nt}^{\mathrm{D}}) < n\left(\sup_{t\geq \bar{\omega}_0^{*,\mathrm{D}}-\xi} \bar{r}_t^{\mathrm{D}} - 2\xi\right)\right].
\end{aligned} \tag{41}$$

First, it is easily seen that

$$\sup_{t\geq \bar{\omega}_0^{*,\mathrm{D}}-\xi} \bar{r}_t^{\mathrm{D}} = \bar{r}^{\mathrm{D}}_{\bar{\omega}_0^{*,\mathrm{D}}-\xi} = \frac{\rho-1}{\rho}\xi < \xi.$$

Therefore:

$$\mathbf{P}\left[\inf_{t\geq \bar{\omega}_0^{*,\mathrm{D}}-\xi} t_1(\nu_{nt}^{\mathrm{D}}) < n\left(\sup_{t\geq \bar{\omega}_0^{*,\mathrm{D}}-\xi} \bar{r}_t^{\mathrm{D}} - 2\xi\right)\right] = 0. \tag{42}$$

On another hand, define the following event:

$$\begin{aligned}
\mathcal{E}_\xi^n := \Bigg\{&\text{for all } t \geq \bar{\omega}_0^{*,\mathrm{D}} - \xi, \text{ some customers arrive before } n(t-\xi), \\
&\qquad\qquad \text{with deadline in } \left[n(t + \bar{r}_t^{\mathrm{D}} + \xi), n(t + \bar{r}_t^{\mathrm{D}} + 2\xi)\right]\Bigg\} \\
= \Bigg\{&\inf_{t\geq \bar{\omega}_0^{*,\mathrm{D}}-\xi} \left\{\tilde{\mathcal{N}}^n_{n(t-\xi),n\left(\bar{r}^{\mathrm{D}}_{t-\xi} + \frac{2\rho+1}{\rho}\xi\right)} - \tilde{\mathcal{N}}^n_{n(t-\xi),n\left(\bar{r}^{\mathrm{D}}_{t-\xi} + \frac{\rho+1}{\rho}\xi\right)}\right\} > 0\Bigg\}.
\end{aligned}$$

We have:

$$
\begin{aligned}
(43)\quad & \mathbf{P}\left[\sup_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left(t_1(\nu_{nt}^{\mathrm{D}})-\bar{r}_t^{\mathrm{D}}\right)>2n\xi\right]\\
& \leq \mathbf{P}\left[\left\{\sup_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left(t_1(\nu_{nt}^{\mathrm{D}})-n\bar{r}_t^{\mathrm{D}}\right)>2n\xi\right\}\right.\\
&\qquad\qquad \left.\cap\left\{\sup_{0\leq t\leq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left|t_1(\nu_{nt}^{\mathrm{D}})-n\bar{r}_t^{\mathrm{D}}\right|\leq n\xi\right\}\cap\mathcal{E}_\xi^n\right]\\
&\qquad + \mathbf{P}\left[\sup_{0\leq t\leq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left|t_1(\bar{\nu}_t^{n,\mathrm{D}})-\bar{r}_t^{\mathrm{D}}\right|>\xi\right]+\mathbf{P}\left[\left(\mathcal{E}_\xi^n\right)^c\right].
\end{aligned}
$$

On the event

$$
\left\{\sup_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left(t_1(\nu_{nt}^{\mathrm{D}})-n\bar{r}_t^{\mathrm{D}}\right)>2n\xi\right\}
\cap\left\{\sup_{0\leq t\leq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left|t_1(\nu_{nt}^{\mathrm{D}})-n\bar{r}_t^{\mathrm{D}}\right|\leq n\xi\right\}\cap\mathcal{E}_\xi^n,
$$

there exists an instant $t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi$ (the first one), such that there is no customer in the system at nt having deadline between $n(t+\bar{r}_t^{\mathrm{D}}+\xi)$ and $n(t+\bar{r}_t^{\mathrm{D}}+2\xi)$. Consider the customer who would have had the smallest time credit at nt if we was still present in the system at this instant, among those arrived before $n(t-\xi)$, with deadline in $[n(t+\bar{r}_t^{\mathrm{D}}+\xi),n(t+\bar{r}_t^{\mathrm{D}}+2\xi)]$. For all $s\leq t-\xi$ such that this customer has already entered the system at ns, denote by $\tilde{R}_s$, the time credit of this customer at ns. We have:

$$
\tilde{R}_s\in\left[n(t-s)+n\bar{r}_t^{\mathrm{D}}+n\xi,n(t-s)+n\bar{r}_t^{\mathrm{D}}+2n\xi\right].
$$

In particular, it is easily seen with the form of $\bar{r}_t^{\mathrm{D}}$ that:

$$
\tilde{R}_s\geq n(t-s)+n\bar{r}_t^{\mathrm{D}}+n\xi>n\bar{r}_s^{\mathrm{D}}+n\xi\geq t_1(\nu_{ns}^{n,\mathrm{D}})
$$

(the last inequality is true since $s\leq\bar{\omega}_0^{*,\mathrm{D}}-\xi$ and

$$
\sup_{0\leq u\leq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left|t_1(\nu_{nu}^{n,\mathrm{D}})-n\bar{r}_u^{\mathrm{D}}\right|\leq n\xi\ ).
$$

Thus, none of the customers arrived before $n(t-\xi)$, with deadline in $[n(t+\bar{r}_t^{\mathrm{D}}+\xi),n(t+\bar{r}_t^{\mathrm{D}}+2\xi)]$ have been served before $n(t-\xi)$, since none of them

has ever been prioritary. Hence:

$$\begin{aligned}(44)\quad &\mathbf{P}\Bigg[\Bigg\{\sup_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left(t_1(\nu^{\mathrm{D}}_{nt})-n\bar{r}^{\mathrm{D}}_t\right)>2n\xi\Bigg\}\\
&\qquad\qquad\cap\Bigg\{\sup_{\varepsilon\leq t\leq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left|t_1(\nu^{\mathrm{D}}_{nt})-n\bar{r}^{\mathrm{D}}_t\right|\leq n\xi\Bigg\}\cap\mathcal{E}^n_\xi\Bigg]\\
&\leq\mathbf{P}\Bigg[\Bigg\{\text{All of the customers arrived before } n(t-\xi),\\
&\qquad\text{with deadline in } [n(t+\bar{r}^{\mathrm{D}}_t+\xi),n(t+\bar{r}^{\mathrm{D}}_t+2\xi)]\\
&\qquad\qquad\text{have entered service between } n(t-\xi) \text{ and } nt\Bigg\}\cap\mathcal{E}^n_\xi\Bigg]\\
&\leq\mathbf{P}\Bigg[\sup_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\Bigg\{S^n_{n\xi}-\Bigg(\tilde{\mathcal{N}}^n_{n(t-\xi),n\left(\bar{r}^{\mathrm{D}}_{t-\xi}+\frac{2\rho+1}{\rho}\xi\right)}\\
&\qquad\qquad-\tilde{\mathcal{N}}^n_{n(t-\xi),n\left(\bar{r}^{\mathrm{D}}_{t-\xi}+\frac{\rho+1}{\rho}\xi\right)}\Bigg)\Bigg\}\geq 0\Bigg],\end{aligned}$$

where $S^n_{n\xi}$ denotes the value at $n\xi$ of a Poisson process of intensity μ^n. The following is a $\mathcal{G}^n_t-$ martingale:

$$\begin{aligned}\check{\mathcal{M}}^n_{t,\xi}:=S^n_{n\xi}-&\left\{\tilde{\mathcal{N}}^n_{n(t-\xi),n\left(\bar{r}^{\mathrm{D}}_{t-\xi}+\frac{2\rho+1}{\rho}\xi\right)}-\tilde{\mathcal{N}}^n_{n(t-\xi),n\left(\bar{r}^{\mathrm{D}}_{t-\xi}+\frac{\rho+1}{\rho}\xi\right)}\right\}\\
&-\left\{\mu^n n\xi-\left(\tilde{\lambda}^n_{(t-\xi),\frac{2\rho+1}{\rho}\xi}-\tilde{\lambda}^n_{(t-\xi),\frac{\rho+1}{\rho}\xi}\right)\right\},\end{aligned}$$

and for all ξ, $\check{\mathcal{M}}^n_{t,\xi}\in L^2$. Then, it is easily seen that the last term of (44) is equal to:

$$\mathbf{P}\left[\sup_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\check{\mathcal{M}}^n_{t,\xi}\geq\inf_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left\{\left(\tilde{\lambda}^n_{(t-\xi),\frac{2\rho+1}{\rho}\xi}-\tilde{\lambda}^n_{(t-\xi),\frac{\rho+1}{\rho}\xi}\right)-\mu^n n\xi\right\}\right].$$

But for all $t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi$:

$$\begin{aligned}&\tilde{\lambda}^n_{(t-\xi),\frac{2\rho+1}{\rho}\xi}-\tilde{\lambda}^n_{(t-\xi),\frac{\rho+1}{\rho}\xi}\\
&=\lambda^n\left\{t-\xi-\left(d-\bar{r}^{\mathrm{D}}_{t-\xi}-\frac{2\rho+1}{\rho}\xi\right)^+\right\}^+\\
&\qquad\qquad-\lambda^n\left\{t-\xi-\left(d-\bar{r}^{\mathrm{D}}_{t-\xi}-\frac{\rho+1}{\rho}\xi\right)^+\right\}^+=\lambda^n\xi,\end{aligned}$$

using (31) and (32). Consequently, for a sufficiently large n:

$$\begin{aligned}(45)\quad &\mathbf{P}\Bigg[\Bigg\{\sup_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left(t_1(\nu_{nt}^{\mathrm{D}})-n\bar{r}_t^{\mathrm{D}}\right)>2n\xi\Bigg\}\\ &\qquad\qquad\cap\Bigg\{\sup_{\varepsilon\leq t\leq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left|t_1(\nu_{nt}^{\mathrm{D}})-n\bar{r}_t^{\mathrm{D}}\right|\leq n\xi\Bigg\}\cap\mathcal{E}_\xi^n\Bigg]\\ &\leq\mathbf{P}\Bigg[\sup_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\check{\mathcal{M}}_{t,\xi}^n\geq n\left(\lambda^n-\mu^n\right)\xi\Bigg]\\ &\qquad\qquad\leq\frac{4}{\left(n\left(\lambda^n-\mu^n\right)\xi\right)^2}\left\{\lambda^n n\xi+\mu^n n\xi\right\}\underset{n\to\infty}{\longrightarrow}0,\end{aligned}$$

using again Doob inequality. Remark, that we also proved that:

$$\begin{aligned}(46)\quad &\mathbf{P}\left[\left(\mathcal{E}_\xi^n\right)^c\right]\\ &=\mathbf{P}\Bigg[\inf_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left(\tilde{\mathcal{N}}^n_{n(t-\xi),n\left(\bar{r}_{t-\xi}^{\mathrm{D}}+\frac{2\rho+1}{\rho}\xi\right)}-\tilde{\mathcal{N}}^n_{n(t-\xi),n\left(\bar{r}_{t-\xi}^{\mathrm{D}}+\frac{\rho+1}{\rho}\xi\right)}\right\}\leq 0\Bigg]\\ &\leq\mathbf{P}\Bigg[\sup_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\Bigg\{S_{n\xi}^n-\Bigg(\tilde{\mathcal{N}}^n_{n(t-\xi),n\left(\bar{r}_{t-\xi}^{\mathrm{D}}+\frac{2\rho+1}{\rho}\xi\right)}\\ &\qquad\qquad-\tilde{\mathcal{N}}^n_{n(t-\xi),n\left(\bar{r}_{t-\xi}^{\mathrm{D}}+\frac{\rho+1}{\rho}\xi\right)}\Bigg)\Bigg\}\geq 0\Bigg]\\ &\qquad\qquad\underset{n\to\infty}{\longrightarrow}0.\end{aligned}$$

Therefore, using (45), (46) and (40) in (43) yields to:

$$\mathbf{P}\Bigg[\sup_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left(t_1(\nu_{nt}^{\mathrm{D}})-\bar{r}_t^{\mathrm{D}}\right)>2n\xi\Bigg]\underset{n\to\infty}{\longrightarrow}0.$$

Together with (42) in (41), this entails:

$$\mathbf{P}\Bigg[\sup_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left|t_1(\bar{\nu}_t^{n,\mathrm{D}})-\bar{r}_t^{\mathrm{D}}\right|>2\xi\Bigg]\underset{n\to\infty}{\longrightarrow}0.$$

This implies in turns, together with (40), that:

$$\begin{aligned}\mathbf{P}\Bigg[\sup_{t\geq 0}\left|t_1(\bar{\nu}_t^{n,\mathrm{D}})-\bar{r}_t^{\mathrm{D}}\right|\geq 3\xi\Bigg]&\leq\mathbf{P}\Bigg[\sup_{0\leq t\leq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left|t_1(\bar{\nu}_t^{n,\mathrm{D}})-\bar{r}_t^{\mathrm{D}}\right|>\xi\Bigg]\\ &\qquad+\mathbf{P}\Bigg[\sup_{t\geq\bar{\omega}_0^{*,\mathrm{D}}-\xi}\left|t_1(\bar{\nu}_t^{n,\mathrm{D}})-\bar{r}_t^{\mathrm{D}}\right|>2\xi\Bigg]\\ &\qquad\underset{n\to\infty}{\longrightarrow}0.\end{aligned}$$

We conclude using [Rob00]. □

We can therefore conclude with the next result, which yields then convergence in distribution of a normalized sequence of profile processes of a M/M/1/1+D-EDF queue to an explicit fluid limit:

Theorem 4 (Fluid limit of the M/M/1/1+D-EDF queue)**.**

$$\left\{\left(\bar{\nu}_t^{n,D}\right)_{t\geq 0}\right\}_{n\in\mathbb{N}^*} \Longrightarrow \left(\bar{\nu}_t^{*,D}\right)_{t\geq 0} \ \text{, } in\ \mathcal{D}\left([0,\infty),\mathcal{M}_f^+\right),$$

where for all $t\geq 0$ and all $f\in\mathcal{D}_b$:

$$\langle \bar{\nu}_t^{*,D}, f\rangle = f(d-t) - \mu\int_0^t f\left(\bar{r}_s^D + s - t\right)\,ds + \lambda\int_0^t f\left(d-s\right)\,ds,$$

$(\bar{r}_t^D)_{t\geq 0}$ *being defined by (29).*

Proof. Let us verify the assumptions of Theorem 3 for any given $T\geq 0$. First, it is straightforward, that Hypothesis 1 are satisfied in view of Hypothesis 2, for $\bar{D}=d$, a.s., and for ξ the Dirac measure at d.

Then, remark, that in (45) we proved as a matter of fact that for all $\xi>0$ satisfying (30) and (32),

$$\mathbf{P}\Bigg[\Bigg\{\text{There exists } t\geq \bar{\omega}_0^{*,\mathrm{D}}-\xi\text{, such that all of the customers arrived before } n(t-\xi)\text{, with deadline in } [n(t+\bar{r}_t^{\mathrm{D}}+\xi), n(t+\bar{r}_t^{\mathrm{D}}+2\xi)] \text{ have been served before } nt\Bigg\}\cap\mathcal{E}_\xi^n\Bigg] \underset{n\to\infty}{\longrightarrow} 0.$$

Denoting this previous event $\mathcal{A}_\xi^n$, it follows that:

$$\mathbf{P}\left[\bar{\tau}_0^{n,\mathrm{D}}\leq T\right]\leq \mathbf{P}\left[\bar{\tau}_0^{n,\mathrm{D}}\leq \bar{\omega}_0^{*,\mathrm{D}}-\xi\right]+\mathbf{P}\left[\mathcal{A}_\xi^n\right]+\mathbf{P}\left[\left(\mathcal{E}_\xi^n\right)^c\right]\underset{n\to\infty}{\longrightarrow}0,$$

using (46) and (39) as well: (21) is satisfied. Finally, (22) is verified in view of Proposition 2 for $(\bar{r}_t^{\mathrm{D}})_{t\geq 0}$ defined in (29). We therefore can apply Theorem 3 for all $T\geq 0$, which completes the proof. □

We can in particular approximate the queue length and loss processes, by applying the profile process of the queue to simple rcll functions. First, in view of (10), the normalized queue length process $\left(\bar{Q}_t^n\right)_{t\geq 0}$ can be asymptotically approximated by the process defined for all $t\geq 0$ by:

$$\begin{aligned}\langle \bar{\nu}_t^{*,\mathrm{D}}, \mathbf{1}_{\mathbb{R}_+^*}\rangle &= -\mu\int_0^t \mathbf{1}_{\mathbb{R}_+^*}\left(\bar{r}_s^{\mathrm{D}}+s-t\right)\,ds+\lambda\int_0^t\mathbf{1}_{\mathbb{R}_+^*}\left(d-s\right)\,ds\\ &= \left\{1+(\lambda-\mu)t\right\}\mathbf{1}_{\left\{t\leq\frac{\rho d-\mu^{-1}}{\rho-1}\right\}}+\lambda d\mathbf{1}_{\left\{t\geq\frac{\rho d-\mu^{-1}}{\rho-1}\right\}}.\end{aligned}$$

Similarly, in view of (11) the normalized loss process $\left(\bar{P}_t^n\right)_{t\geq 0}$ can be approximated by the process defined for all $t\geq 0$ by:

$$\begin{aligned}\langle \bar{\nu}_t^{*,\mathrm{D}}, \mathbf{1}_{\mathbb{R}_-}\rangle &= -\mu\int_0^t \mathbf{1}_{\mathbb{R}_-}\left(\bar{r}_s^{\mathrm{D}}+s-t\right)\,ds+\lambda\int_0^t\mathbf{1}_{\mathbb{R}_-}\left(d-s\right)\,ds\\ &= \left(1+\lambda(t-d)-\mu t\right)\mathbf{1}_{\left\{t\geq\frac{\rho d-\mu^{-1}}{\rho-1}\right\}}.\end{aligned}$$

M/GI/∞ system. Remark, that we can also apply Theorem 3 to obtain the fluid limit of a pure delay M/GI/∞ system. For all $n \in \mathbb{N}^*$, consider a pure delay (PD) M/GI/∞ system: customers arrive according to a Poisson process of intensity λ^n, requesting service durations, i.i.d. of the distribution of α^n, to an infinite reservoir of servers, assuming that:

$$\lambda^n \to \lambda,\ \mathbf{E}\left[\frac{\alpha^n}{n}\right] \to \mathbf{E}\left[\alpha\right],\ \frac{\alpha^n}{n} \xrightarrow{\mathcal{D}} \alpha.$$

Each customer is hence immediately attended upon arrival, and remains in the system for the duration of his service. Such a system can be described by a profile process, keeping now track of the remaining processing times of the customers in service (positive atoms) and the elapsed times since departure of the already served customers (negative atoms): this is the profile process $\left(\nu_t^{n,\mathrm{PD}}\right)_{t\geq 0}$ associated to $\left(\nu_0^{n,\mathrm{PD}}, \lambda^n, 0, \alpha^n\right)$, where $\nu_0^{n,\mathrm{PD}}$ is the profile of the service durations of the customers initially in the system (which we assume to be an n-sample of the distribution of α^n). The service durations replace the time credits of the queue with impatient customers, and consequently the analogous service rate becomes null in this case. We normalize this process as above, writing for all $t \geq 0$ and all Borel set $\mathfrak{B}$:

$$\bar{\nu}_t^{n,\mathrm{PD}}(\mathfrak{B}) = \frac{\nu_{nt}^{n,\mathrm{PD}}(n\mathfrak{B})}{n}.$$

The fact that μ^n is zero allows us to skip conditions (21) and (22) in Theorem 3, whose application becomes straightforward:

$$\left\{\left(\bar{\nu}_t^{n,\mathrm{PD}}\right)_{t\geq 0}\right\}_{n\in\mathbb{N}^*} \Longrightarrow \left(\bar{\nu}_t^{*,\mathrm{PD}}\right)_{t\geq 0} \text{ in } \mathcal{D}\left([0,\infty), \mathcal{M}_f^+\right),$$

where $\left(\bar{\nu}_t^{*,\mathrm{PD}}\right)_{t\geq 0}$ is the deterministic element of $\mathcal{D}_{\infty,\mathcal{M}_f^+}$ defined for all $t \geq 0$ and all $f \in \mathcal{D}_b$ by:

$$\langle \bar{\nu}_t^{*,\mathrm{PD}}, f\rangle = \mathbf{E}\left[\tau_t f(\alpha)\right] + \lambda \int_0^t \mathbf{E}\left[\tau_s f(\alpha)\right]\, ds.$$

Hence, as above, we can asymptotically approximate the normalized congestion process (number of customers in service), given for all $n \in \mathbb{N}^*$ by $\left(\langle \bar{\nu}_t^{n,\mathrm{PD}}, \mathbf{1}_{\mathbb{R}_+^*}\rangle\right)_{t\geq 0}$ by the process defined for all $t \geq 0$ by:

$$\langle \bar{\nu}_t^{*,\mathrm{PD}}, \mathbf{1}_{\mathbb{R}_+^*}\rangle = \mathbf{P}\left[\alpha > t\right] + \lambda \int_0^t \mathbf{P}\left[\alpha > s\right]\, ds.$$

Remark, that in the special case where α^n is exponentially distributed of parameter $\mu^n \to \mu$, this process becomes for all $t \geq 0$:

$$e^{-\mu t} + \rho\left(1 - e^{-\mu t}\right),$$

which is the fluid limit obtained by Borovkov in [Bor67]. We can also approximate the normalized workload process, given for all n by $\left(\langle \bar{\nu}_t^{n,\mathrm{PD}}, I\mathbf{1}_{\mathbb{R}_+^*}\rangle\right)_{t\geq 0}$ by the process given by:

$$\langle \bar{\nu}_t^{*,\mathrm{PD}}, I\mathbf{1}_{\mathbb{R}_+^*}\rangle = \mathbf{E}\left[(\alpha - t)^+\right] + \lambda \int_0^t \mathbf{E}\left[(\alpha - s)^+\right]\, ds.$$

Finally, the normalized process counting the already served customers can be approximated by the process defined for all $t \geq 0$ by:

$$\langle \bar{\nu}_t^{*,\mathrm{PD}}, \mathbf{1}_{\mathbb{R}-} \rangle = \mathbf{P}\left[\alpha \leq t\right] + \lambda \int_0^t \mathbf{P}\left[\alpha \leq s\right] ds.$$

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